

Non-geometric Fluxes, Asymmetric Strings and Nonassociative Geometry

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Abstract

We study closed bosonic strings propagating both in a flat background with constant H -flux and in its T-dual configurations. We define a conformal field theory capturing linear effects in the flux and compute scattering amplitudes of tachyons, where the Rogers dilogarithm plays a prominent role. For the scattering of four tachyons, a fluxed version of the Virasoro-Shapiro amplitude is derived and its pole structure is analyzed. In the case of an R -flux background obtained after three T-dualities, we find indications for a nonassociative target-space structure which can be described in terms of a deformed tri-product. Remarkably, this product is compatible with crossing symmetry of conformal correlation functions. We finally argue that the R -flux background flows to an asymmetric CFT.

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1 Introduction

The basic space-time principles underlying string theory are still elusive to a large extent. Indeed, one of the important differences between a string and a point particle is how they probe the space-time they are moving in. For instance, due to the decoupling of the left- and right-moving sectors, strings can not only propagate in left-right symmetric and thus geometric backgrounds, but also in backgrounds defined via a left-right asymmetric conformal field theory. Examples are asymmetric orbifolds, whose target-space interpretation is not clear a priori. Note that for point particles, this situation does not occur. Related to this feature is the observation that strings cannot distinguish between spaces which are related by T-duality, which from the world-sheet point of view is a left-right asymmetric operation.

In the past, applying T-duality to known configurations has led to new insight into string theory, where a prominent example is the discovery of D-branes. Furthermore, in [1, 2] T-duality was applied to a simple closed string background, namely to a flat space with non-vanishing three-form flux $H = dB$, which resulted in a background with geometric flux. This so-called twisted torus is still a conventional string background, but a second T-duality leads to a non-geometric flux background. These are spaces in which the transition functions between two charts of a manifold are allowed to be T-duality transformations, hence they are also called T-folds [3, 4, 5]. After formally applying a third T-duality, not along an isometry direction any more, one obtains an R -flux background which does not admit a clear target-space interpretation. It was proposed not to correspond to an ordinary geometry even locally, but instead to give rise to a nonassociative geometry [6, 7] (see also [8, 9]).

From another point of view, the general expectation is that in quantum gravity a noncommutative structure of space-time emerges. In string theory, noncommutativity can be seen for open strings whose end-points probe a D-brane endowed with a non-trivial two-form background. In this case, it is a well-established result that the coordinates on the D-brane do not commute, that is $[x^a, x^b] = \theta^{ab}$. Moreover, it has been shown explicitly in [10] that one can reformulate the theory on the D-brane in terms of a noncommutative gauge theory, where the ordinary product of two fields is deformed to the Moyal-Weyl product. This structure can be tested directly by the evaluation of on-shell open string scattering amplitudes. The question now is whether a similar noncommutative space-time structure also appears for closed strings, i.e. for those objects which correspond to gravity.

In [11] it was argued that, since the world-sheet of a closed string is a sphere, in this case not two but three coordinates should be involved. For the well-understood background of the $SU(2)_k$ WZW model, the world-sheet equal-time, equal-space Jacobi-identity for three space-time coordinates was evaluated and found to be non-vanishing. This gave rise to the conjecture that the space-time coordinates satisfy a non-trivial three-bracket relation $[x^a, x^b, x^c] = \theta^{abc}$, where θ^{abc} is related to the three-form flux. Thus, the space-time coordinates are not just noncommutative but also nonassociative, and we call such geometries noncommutative/nonassociative (NCA). Let us note that an analogous structure also appears for an open string ending on a D-brane with constant H -flux, which was demonstrated in [12, 13, 14].

In a complementary approach [15], the commutation relations between coordinates of three-dimensional backgrounds with geometric as well as with T-dual non-geometric flux were studied. It was found that for non-geometric flux the commutator between two coordinates x^a and x^b is non-vanishing, and in the case of geometric flux the commutator between x^a and one dual coordinate \tilde{x}^b is non-zero (see [15] for more details). Studying then the canonical commutation relations between (dual) coordinates and (dual) momenta, one is again led to a nonassociative algebra.

In this paper, we study some non-geometric aspects of string theory, in particular asymmetric string solutions, non-geometric fluxes and nonassociative geometry. Similar to [1, 2], our starting point is that of a closed bosonic string moving through a flat background with constant H -flux. From the string equations of motion one infers that this configuration is not conformal beyond linear order in the flux. However, since the three-bracket relation mentioned above is linear in the flux parameter θ^{abc} , we expect to see NCA effects already at this order. We then proceed in the following way. In section 2, we briefly review some results of [10] concerning open string correlators and the derivation of the noncommutative Moyal-Weyl product, which we want to generalize to the closed string sector.

In section 3 we construct the conformal field theory corresponding to the H -flux background at linear order. This involves two effects: first, the coordinates and related currents are redefined at linear order in H , and second that correlation functions and operator product expansions (OPEs) receive corrections which we evaluate using conformal perturbation theory. We compute the basic three-point function $\langle \mathcal{X}^a \mathcal{X}^b \mathcal{X}^c \rangle$ of three coordinates \mathcal{X}^a , which turns out to be proportional to the three-form flux. We then introduce vertex operators for the deformed theory and analyze whether they still correspond to physical states. Here, logarithmic terms appear in OPEs and correlation functions. Furthermore, we analyze T-dual configurations and illustrate that one can draw direct conclusions on the underlying target-space structure only for the two cases of pure momentum-mode scattering in the H -flux and in the R -flux background.

Section 4 is devoted to the computation of tachyon scattering amplitudes. We find that tachyon amplitudes are crossing symmetric after momentum conservation has been employed. Furthermore, in case of R -flux, we detect non-trivial phase factors which can be encoded in a new nonassociative tri-product. Then, we analyze the four-tachyon scattering amplitude in more detail and derive a conformally invariant, crossing symmetric fluxed version of the Virasoro-Shapiro amplitude. Like the Veneziano or the usual Virasoro-Shapiro amplitude, its pole structure reveals information on the spectrum and couplings of the full theory. We find new tachyons, indicating instabilities due to higher order corrections in the flux.

In section 5 we speculate how these instabilities lead to new backgrounds after tachyon condensation. In particular, we make a proposal for the case of R -flux which involves a left-right asymmetric WZW-type background. We also generalize the noncommutative Moyal-Weyl product to a nonassociative tri-product. Finally, in three appendices we provide additional details on the Rogers dilogarithm, the (non-)geometry of the T-dual backgrounds and on the computation of scattering amplitudes.

Note added in proof: upon finishing this manuscript, a paper with partial overlap compared to our work has appeared [16], which discusses correlation functions and T-duality for backgrounds with geometric flux.

2 Open and closed strings in flux backgrounds

In this paper, we study closed strings in backgrounds with constant H -flux as well as in its T-dual configurations. Due to the back-reaction of the flux on the metric, this task is rather difficult. However, for open strings ending on a D-brane with constant two-form background one can analyze the system via methods of two-dimensional conformal field theory. In this section, we therefore first recall some basic facts from the open string story, before applying similar techniques to the closed string sector in the sequel.

2.1 The Moyal-Weyl product for open strings

We start by reviewing some aspects of the work of Seiberg and Witten [10] about noncommutativity for open strings [17, 18]. In particular, we consider open strings ending on a D-brane carrying a non-trivial two-form flux $\mathcal{F} = B + F$. In this case, the two-point function of two open string coordinates $X^a(z)$ inserted on the boundary of a disc takes the form [19, 20, 21, 10]

$$\langle X^a(\tau_1) X^b(\tau_2) \rangle = -\alpha' G^{ab} \log(\tau_1 - \tau_2)^2 + i \theta^{ab} \epsilon(\tau_1 - \tau_2) , \quad (2.1)$$

where τ stands for the real part of the complex world-sheet coordinate z . The matrix G^{ab} is symmetric and can be interpreted as the (inverse of the) effective metric seen by the open string. The matrix θ^{ab} is proportional to the two-form flux \mathcal{F} and thus is anti-symmetric, and it can be interpreted as a noncommutativity parameter. The function $\epsilon(\tau)$ is defined as

$$\epsilon(\tau) = \begin{cases} +1 & \tau \geq 0 , \\ -1 & \tau < 0 , \end{cases} \quad (2.2)$$

and it is the appearance of the jump given by $\epsilon(\tau_1 - \tau_2)$ in (2.1) which leads to noncommutativity of the open string coordinates on the D-brane.

Next, we recall the form of open string vertex operators which are inserted at the boundary of a disc diagram. Employing the short-hand notation $p \cdot X = p_a X^a$ and denoting normal ordered products by $:\dots:$, a tachyon vertex operator can be written as

$$T \equiv V_p(\tau) = : \exp(ip \cdot X(\tau)) : . \quad (2.3)$$

A correlation function of N such vertex operators is found to be

$$\langle T_1 \dots T_N \rangle = \exp \left(i \sum_{1 \leq n < m \leq N} p_{n,a} \theta^{ab} p_{m,b} \epsilon(\tau_n - \tau_m) \right) \times \langle T_1 \dots T_N \rangle_{\theta=0} , \quad (2.4)$$

which contains an extra phase due to the noncommutative nature of the theory. Note that because of momentum conservation $\sum_{n=1}^N p_n = 0$, this correlator is invariant under cyclic permutations of the N vertex operators. Therefore, since a

conformal $SL(2, \mathbb{R})$ transformation can only induce cyclic permutations of points along the real axis, the correlation function (2.4) is $SL(2, \mathbb{R})$ invariant. One can then define an N -product \star_N in the following way

$$f_1(x) \star_N f_2(x) \star_N \dots \star_N f_N(x) := \exp \left(i \sum_{1 \leq n < m \leq N} \theta^{ab} \partial_a^{x_n} \partial_b^{x_m} \right) f_1(x_1) f_2(x_2) \dots f_N(x_N) \Big|_{x_1 = \dots = x_N = x}, \quad (2.5)$$

which correctly reproduces the phase appearing in (2.4). Note that these N -products are related to the subsequent application of the usual star-product $\star = \star_2$

$$f_1 \star_N f_2 \star_N \dots \star_N f_N = f_1 \star f_2 \star \dots \star f_N, \quad (2.6)$$

and therefore, by evaluating correlation functions of vertex operators in open string theory, it is possible to derive the Moyal-Weyl product and some of its features. In the following sections, we apply this approach to correlation functions in the closed string sector.

Let us also recall how to compute correlators of massless states such as gluons. Their (integrated) on-shell vertex operator is given by

$$\mathcal{A} \equiv \int d\tau V_\xi(\tau) = \int d\tau \xi \cdot \partial_\tau X \exp(i p \cdot X(\tau)), \quad (2.7)$$

subject to the restrictions $p^2 = 0$ and $\xi \cdot p = 0$. The correlation function of three gluons for a given order of insertions of operators along the real axis can be computed as

$$\begin{aligned} \langle \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \rangle = & \left[(\xi_1 \cdot \xi_2)(p_2 \cdot \xi_3) + (\xi_1 \cdot \xi_3)(p_1 \cdot \xi_2) + (\xi_2 \cdot \xi_3)(p_3 \cdot \xi_1) \right. \\ & \left. + 2\alpha' (p_1 \cdot \xi_2)(p_2 \cdot \xi_3)(p_3 \cdot \xi_1) \right] \exp \left(i p_{1,a} \theta^{ab} p_{2,b} \right) \delta(p_1 + p_2 + p_3), \end{aligned} \quad (2.8)$$

where momentum conservation has been used in the phase factor. Let us emphasize that from (2.8) we see that the noncommutative product directly effects gluon scattering amplitudes, and therefore will modify the corresponding low-energy effective action.

2.2 Closed strings in flux backgrounds

In the previous subsection, we have illustrated how to derive the noncommutative Moyal-Weyl product from correlation functions in open string theory, and we have pointed out that this product effects scattering amplitudes and therefore the low energy effective description of the gauge theory. The question we would like to ask in this paper is whether this intriguing structure also appears in closed string theory.

T-dualized directions	—	(x^3)	(x^2, x^3)	(x^1, x^2, x^3)
T-dual flux	H -flux	ω -flux	Q -flux	R -flux
momentum/winding	—	$(p_3 \leftrightarrow w_3)$	$\begin{pmatrix} p_2 \leftrightarrow w_2 \\ p_3 \leftrightarrow w_3 \end{pmatrix}$	$\begin{pmatrix} p_1 \leftrightarrow w_1 \\ p_2 \leftrightarrow w_2 \\ p_3 \leftrightarrow w_3 \end{pmatrix}$

Table 1: T-dualities and their action on the flux and on momentum/winding states.

One of the main differences between the open and closed string in this respect is that for the latter a constant two-form potential B can be gauged away and the relevant flux background is expected to be given by the three-form $H = dB$.¹ However, the string equations of motion imply that a non-vanishing H -flux back-reacts on the metric so that a flat background endowed with flux does not correspond to a two-dimensional CFT on the string world-sheet. On the other hand, as we will review below, the back-reaction appears only at second order in the flux and we can expect a bona-fide conformal field theory up to linear order in H . We denote this theory by CFT_H which we study in detail in section 3.

Let us also comment on configurations T-dual to the above background and consider a three-dimensional compact space with constant H -flux and vanishing curvature. Following the same spirit as in [1], we can T-dualize this background along an arbitrary number of directions which is summarized in table 1. There, we have shown how momentum and winding modes are exchanged (the mapping for the Q - and R -flux has only been obtained by generalizing the ω -flux result), and more details on these T-dual backgrounds are collected in appendix A.1. From table 1 we see that a first T-duality, say along the x^3 -direction, maps the H -flux to a geometric flux ω which in the compact case defines a twisted torus. After a second T-duality one obtains a so-called T-fold [3, 4, 5] defined via a non-geometric Q -flux. The resulting T-fold actually does not any longer posses an isometry direction so that it is not clear whether a formal T-duality along a third direction is allowed. It was argued that the resulting R -flux background is not even locally an ordinary space, but rather gives rise to a nonassociative geometry [6, 7].

3 Conformal field theory with H -flux

In this section, we approach the question whether a noncommutative structure analogous to Seiberg/Witten can be obtained for the closed string. To do so, we first recall some previous results on that matter and specify the framework for this paper. We then determine the correlator of three closed string coordinates as well as corrections to the two-point function due to a background H -flux. Finally, we introduce vertex operators for the perturbed theory and discuss features thereof.

¹Open strings in such an H -flux background have been studied for instance in [22, 12, 13, 14].

3.1 Prerequisites

The motivation for our study is the recent paper [11] where the question mentioned above was discussed for the $SU(2)_k$ WZW model. This model describes a string moving on S^3 with flux through the sphere, where the radius of S^3 is related to the flux such that the string equations of motion (corresponding to conformal symmetry on the world-sheet) are satisfied to all orders in sigma-model perturbation theory. For this background, the equal-time cyclic double-commutator of three local coordinates was found to be [11]

$$\lim_{z_i \rightarrow z} \left[X^a(z_1, \bar{z}_1), [X^b(z_2, \bar{z}_2), X^c(z_3, \bar{z}_3)] \right] + \text{cycl.} = \begin{cases} \epsilon \theta^{abc} & z_i \rightarrow z, \\ 0 & \text{else,} \end{cases} \quad (3.1)$$

where $\theta^{abc} \sim H^{abc}$ encodes the three-form flux and where the parameter ϵ turns out to be $\epsilon = 0$ for the H -flux background and $\epsilon = 1$ for the background one obtains after an odd number of T-dualities. Note that the zero-mode contribution in the above computation was fixed by the requirement that the Jacobi-identity vanishes if some of the z_i are not equal (see [11] for more details). In particular, a constant term which does not depend on z_i is absent in (3.1). One can then define and compute

$$[X^a, X^b, X^c] := \lim_{z_i \rightarrow z} \left[X^a(z_1, \bar{z}_1), [X^b(z_2, \bar{z}_2), X^c(z_3, \bar{z}_3)] \right] + \text{cycl.} = \epsilon \theta^{abc}. \quad (3.2)$$

Thus, in the case of R -flux with $\epsilon = 1$, the space-time coordinates X^a satisfy a non-vanishing three-bracket,² and therefore may give rise to a nonassociative geometry.

H -flux background

In this paper, we do not start from an exactly solvable WZW model and then take a local limit, but rather from a flat background with constant H -flux. This approach is analogous to that for the open string in a constant B -field background, with the main difference that here we cannot decouple gravity and so the back-reaction of the flux has to be considered.

More concretely, our framework is that of a flat space with constant H -flux and dilaton which is to be considered as part of a full bosonic string theory construction. The metric and the flux are specified by

$$ds^2 = \sum_{a=1}^N (dX^a)^2, \quad H = \frac{2}{\alpha'^2} \theta_{abc} dX^a \wedge dX^b \wedge dX^c, \quad (3.3)$$

where in the following we focus mostly on $N = 3$, but our discussion can be readily generalized. Note that already at lowest order in α' , this background is

²A noncommutative quantum field theory based on a similar three-product has been proposed in [23].

not a solution to the string equations of motion. In particular, the beta-functional for the graviton

$$\beta_{ab}^G = \alpha' R_{ab} - \frac{\alpha'}{4} H_a{}^{cd} H_{bcd} + 2\alpha' \nabla_a \nabla_b \Phi + O(\alpha'^2) \quad (3.4)$$

does not vanish for (3.3) in the case of a constant dilaton Φ . Only at *linear order* in the H -flux the above background provides a solution, and by dimensional analysis it is clear that at higher orders in α' there can be no further obstructions at linear order in H . We can thus conclude that the flat space background with constant H and Φ corresponds to a bona fide conformal field theory at linear order in the flux. Furthermore, since the three-bracket (3.1) is linear in $\theta^{abc} \sim H^{abc}$, up to first order in the H -flux we expect to find a reliable world-sheet CFT framework capturing potential nonassociative effects.

Now, a closed string moving in the background given by (3.3) can be described by a sigma-model. With Σ denoting the world-sheet of the closed string, its action reads

$$\mathcal{S} = \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2z \left(g_{ab} + B_{ab} \right) \partial X^a \bar{\partial} X^b, \quad (3.5)$$

where the metric for our particular situation is given by $g_{ab} = \delta_{ab}$. For the B -field we can choose a gauge in which $B_{ab} = \frac{1}{3} H_{abc} X^c$; and the equations of motion are of the form

$$\partial \bar{\partial} X^a = \frac{1}{2} H^a{}_{bc} \partial X^b \bar{\partial} X^c. \quad (3.6)$$

At zero order in H , the solution to (3.6) is that of the well-known free theory for which we employ the notation

$$\mathbf{X}_0^a(z, \bar{z}) = \mathbf{X}_L^a(z) + \mathbf{X}_R^a(\bar{z}), \quad (3.7)$$

where we made a distinction between the solution \mathbf{X}^a and the field X^a . At linear order in the flux, a solution to (3.6) is given by

$$\mathbf{X}_1^a(z, \bar{z}) = \mathbf{X}_0^a(z, \bar{z}) + \frac{1}{2} H^a{}_{bc} \mathbf{X}_L^b(z) \mathbf{X}_R^c(\bar{z}). \quad (3.8)$$

Perturbation theory

The natural approach to compute the correlation functions in the above setting is conformal perturbation theory [24, 19, 25, 26, 21, 20, 27, 12]. Writing the action (3.5) as the sum of a free part \mathcal{S}_0 and a perturbation \mathcal{S}_1 , and choosing again a gauge in which $B_{ab} = \frac{1}{3} H_{abc} X^c$, we have

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1 \quad \text{where} \quad \mathcal{S}_1 = \frac{1}{2\pi\alpha'} \frac{H_{abc}}{3} \int_{\Sigma} d^2z X^a \partial X^b \bar{\partial} X^c. \quad (3.9)$$

Note that we expect \mathcal{S}_1 to be a marginal operator, but not exactly marginal since it will be obstructed already at second order in H . We will find evidence for this fact in the following.

A correlation function of N operators $\mathcal{O}_i[X]$ can be computed via the path integral in the usual way

$$\langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle = \frac{1}{\mathcal{Z}} \int [dX] \mathcal{O}_1 \dots \mathcal{O}_N e^{-S[X]}, \quad (3.10)$$

where \mathcal{Z} denotes the vacuum functional given by $\mathcal{Z} = \int [dX] e^{-S[X]}$. In the limit of small fluxes, it is possible to expand (3.10) in the perturbation \mathcal{S}_1 leading to

$$\begin{aligned} \langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle &= \langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle_0 - \langle \mathcal{O}_1 \dots \mathcal{O}_N \mathcal{S}_1 \rangle_0 \\ &+ \frac{1}{2} \left[\langle \mathcal{O}_1 \dots \mathcal{O}_N \mathcal{S}_1^2 \rangle_0 - \langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle_0 \times \langle \mathcal{S}_1^2 \rangle_0 \right] + \mathcal{O}(H^3), \end{aligned} \quad (3.11)$$

where for later purpose we have included corrections up to second order in H . The subscript 0 indicates that the correlator is computed using the free action \mathcal{S}_0 , and we made use of the fact that $\langle \mathcal{S}_1 \rangle_0$ vanishes. The latter can be verified using the two-point function of two free fields $X^a(z, \bar{z})$ (as well as derivatives thereof)

$$\langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) \rangle_0 = -\frac{\alpha'}{2} \log |z_1 - z_2|^2 \delta^{ab}. \quad (3.12)$$

3.2 Three- and two-point function

In this section, we determine the correlator of three closed string coordinates $\mathcal{X}^a(z, \bar{z})$ and the correction to the two-point function (3.12) up to second order in the flux parameter.

Three-current correlators

Let us start by noting that the fields $X^a(z, \bar{z})$ appearing in the action (3.5) are actually not proper conformal fields of the theory. Only the currents have a well-defined behavior under conformal transformations. Therefore, as usual, for the free theory we define

$$J^a(z) = i\partial X^a(z), \quad \bar{J}^a(\bar{z}) = i\bar{\partial} X^a(z), \quad (3.13)$$

which at zeroth order in H are indeed holomorphic and anti-holomorphic, respectively. The correlator of three say holomorphic currents $J^a(z)$ up to first order in the H -flux is then computed using (3.11) as follows

$$\begin{aligned} &\langle J^a(z_1) J^b(z_2) J^c(z_3) \rangle \\ &= -\langle J^a(z_1) J^b(z_2) J^c(z_3) \mathcal{S}_1 \rangle_0 \\ &= -\frac{H_{pqr}}{6\pi\alpha'} \int_{\Sigma} d^2w \langle J^a(z_1) J^b(z_2) J^c(z_3) X_0^p(\omega, \bar{\omega}) \partial X_0^q(\omega) \bar{\partial} X_0^r(\bar{\omega}) \rangle_0. \end{aligned} \quad (3.14)$$

The expression in the last line can be evaluated via Wick's theorem employing the two-point function (3.12) as well as

$$\partial_{z_1} \bar{\partial}_{z_2} \log |z_1 - z_2|^2 = -2\pi \delta^{(2)}(z_1 - z_2) . \quad (3.15)$$

Taking into account the antisymmetry of H_{abc} , raising the indices of H with δ^{ab} and using $z_{ij} = z_i - z_j$, for the correlators of three currents (3.13) (up to first order in the H -flux) we find

$$\begin{aligned} \langle J^a(z_1) J^b(z_2) J^c(z_3) \rangle &= -i \frac{\alpha'^2}{8} H^{abc} \frac{1}{z_{12} z_{23} z_{13}} , \\ \langle J^a(z_1) J^b(z_2) \bar{J}^c(\bar{z}_3) \rangle &= -i \frac{\alpha'^2}{8} H^{abc} \frac{\bar{z}_{12}}{z_{12}^2 \bar{z}_{23} \bar{z}_{13}} , \\ \langle \bar{J}^a(\bar{z}_1) \bar{J}^b(\bar{z}_2) J^c(z_3) \rangle &= +i \frac{\alpha'^2}{8} H^{abc} \frac{z_{12}}{\bar{z}_{12}^2 z_{23} z_{13}} , \\ \langle \bar{J}^a(\bar{z}_1) \bar{J}^b(\bar{z}_2) \bar{J}^c(\bar{z}_3) \rangle &= +i \frac{\alpha'^2}{8} H^{abc} \frac{1}{\bar{z}_{12} \bar{z}_{23} \bar{z}_{13}} . \end{aligned} \quad (3.16)$$

As one can see, these expressions are not purely holomorphic or purely anti-holomorphic, but mixed terms appear. However, we have been using the currents (3.13) which are only valid for the free theory. To work at first order in the flux, we should take into account corrections to (3.13) linear in H . Let us therefore define new fields \mathcal{J}^a and $\bar{\mathcal{J}}^a$ in terms of (3.13) in the following way

$$\begin{aligned} \mathcal{J}^a(z, \bar{z}) &= J^a(z) - \frac{1}{2} H^a_{bc} J^b(z) X_R^c(\bar{z}) , \\ \bar{\mathcal{J}}^a(z, \bar{z}) &= \bar{J}^a(\bar{z}) - \frac{1}{2} H^a_{bc} X_L^b(z) \bar{J}^c(\bar{z}) . \end{aligned} \quad (3.17)$$

For the fields (3.17), the only non-vanishing correlators of three fields (up to first order in the flux) are then either purely holomorphic or purely anti-holomorphic

$$\begin{aligned} \langle \mathcal{J}^a(z_1, \bar{z}_1) \mathcal{J}^b(z_2, \bar{z}_2) \mathcal{J}^c(z_3, \bar{z}_3) \rangle &= -i \frac{\alpha'^2}{8} H^{abc} \frac{1}{z_{12} z_{23} z_{13}} , \\ \langle \bar{\mathcal{J}}^a(z_1, \bar{z}_1) \bar{\mathcal{J}}^b(z_2, \bar{z}_2) \bar{\mathcal{J}}^c(z_3, \bar{z}_3) \rangle &= +i \frac{\alpha'^2}{8} H^{abc} \frac{1}{\bar{z}_{12} \bar{z}_{23} \bar{z}_{13}} . \end{aligned} \quad (3.18)$$

Thus, the corrected fields $\mathcal{J}^a(z, \bar{z})$ and $\bar{\mathcal{J}}^a(z, \bar{z})$, respectively, have holomorphic and anti-holomorphic correlation functions. Furthermore, using the equation of motion (3.6), we compute

$$\bar{\partial} \mathcal{J}^a(z, \bar{z}) = 0 , \quad \partial \bar{\mathcal{J}}^a(z, \bar{z}) = 0 , \quad (3.19)$$

so these fields are indeed holomorphic and anti-holomorphic, and from now on will be denoted as $\mathcal{J}^a(z)$ and $\bar{\mathcal{J}}^a(\bar{z})$. Note also that the correlators (3.18) agree with the three-point function of three currents in the $SU(2)_k$ WZW-model, up to an already anticipated [11] sign change for the “structure constants” H^{ab}_c .

Basic three-point function

Let us next define fields $\mathcal{X}^a(z, \bar{z})$ as the integrals of (3.17). In particular, we write

$$\mathcal{J}^a(z) = i\partial\mathcal{X}^a(z, \bar{z}) , \quad \bar{\mathcal{J}}^a(\bar{z}) = i\bar{\partial}\mathcal{X}^a(z, \bar{z}) . \quad (3.20)$$

The three-point function of three \mathcal{X}^a up to first order in the H -flux can then be obtained by integrating the corresponding correlators (3.18). For that purpose, we introduce the Rogers dilogarithm $L(z)$ which is defined in terms of the usual dilogarithm $\text{Li}_2(z)$ as follows

$$L(z) = \text{Li}_2(z) + \frac{1}{2} \log(z) \log(1-z) . \quad (3.21)$$

In appendix A.2, we have collected some useful properties of this function. For the correlator of three fields (3.8) one obtains³

$$\begin{aligned} & \langle \mathcal{X}^a(z_1, \bar{z}_1) \mathcal{X}^b(z_2, \bar{z}_2) \mathcal{X}^c(z_3, \bar{z}_3) \rangle \\ &= \frac{\alpha'^2}{12} H^{abc} \left[L\left(\frac{z_{12}}{z_{13}}\right) + L\left(\frac{z_{23}}{z_{21}}\right) + L\left(\frac{z_{13}}{z_{23}}\right) - \text{c.c.} \right] + F , \end{aligned} \quad (3.22)$$

where “c.c.” stands for complex conjugation and where we have included integration constants collectively denoted by F . These have to satisfy $\partial_i \partial_j \partial_k F = 0$ with $i \in \{z_1, \bar{z}_1\}$, $j \in \{z_2, \bar{z}_2\}$, $k \in \{z_3, \bar{z}_3\}$.

Let us note that analogous terms can appear for the two-point function of two fields $X^a(z, \bar{z})$. Indeed, (3.12) is actually not well-defined on a two-sphere; only the two-point functions of the associated currents (3.13) are bona-fide conformal objects. In particular, one has the freedom to add additional terms $f(z, \bar{z})$ satisfying $\partial_i \partial_j f = 0$, where $i \in \{z_1, \bar{z}_1\}$ and $j \in \{z_2, \bar{z}_2\}$

$$\langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) \rangle_0 = -\frac{\alpha'}{2} \left(\log |z_1 - z_2|^2 + f(z_1, \bar{z}_1) + f(z_2, \bar{z}_2) \right) \delta^{ab} . \quad (3.23)$$

However, these extra terms do not contribute to physical amplitudes and for this reason we have not included them in (3.12) in the first place. Furthermore, for the two-point function on the sphere one can show explicitly that the amplitudes of vertex operators are indeed independent of the extra terms by introducing a background charge. For (3.22) it would be interesting to have a similar proof; here we first proceed with the assumption that $F = 0$. Later, for the mathematically correct derivation of the four-tachyon amplitude these terms will become relevant.

To simplify our notation for the following, let us recall from (3.3) the relation between the flux H and the flux parameter θ , that is $\theta^{abc} = \frac{\alpha'^2}{12} H^{abc}$, and let us introduce

$$\mathcal{L}(z) = L(z) + L\left(1 - \frac{1}{z}\right) + L\left(\frac{1}{1-z}\right) . \quad (3.24)$$

³The Rogers dilogarithm also appeared in the analogous open string three-point function discussed in [12].

Note that this sum of dilogarithms satisfies the relations

$$\mathcal{L}(z) = \mathcal{L}\left(1 - \frac{1}{z}\right) = \mathcal{L}\left(\frac{1}{1-z}\right) , \quad \mathcal{L}(z) + \mathcal{L}(1-z) = 3L(1) = \frac{\pi^2}{2} . \quad (3.25)$$

The correlation function (3.22) of three fields $\mathcal{X}^a(z, \bar{z})$ in the H -flux background can then be written as

$$\langle \mathcal{X}^a(z_1, \bar{z}_1) \mathcal{X}^b(z_2, \bar{z}_2) \mathcal{X}^c(z_3, \bar{z}_3) \rangle = \theta^{abc} \left[\mathcal{L}\left(\frac{z_{12}}{z_{13}}\right) - \mathcal{L}\left(\frac{\bar{z}_{12}}{\bar{z}_{13}}\right) \right] . \quad (3.26)$$

Correction to the two-point function

In our above discussion, we have seen that at linear order in the H -flux, the three-point function for the corrected fields \mathcal{X}^a is conformally invariant. However, at second order this is no longer true which we want to illustrate for the two-point function.

For the regular fields $X^a(z, \bar{z})$ the two-point function up to second order in the flux can be computed using formula (3.11). Since the correction at linear order in H vanishes for a correlator of two fields, we are left with expressions quadratic in the perturbation \mathcal{S}_1

$$\begin{aligned} \delta_2 \langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) \rangle &= + \frac{1}{2} \langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) \mathcal{S}_1^2 \rangle_0 \\ &\quad - \frac{1}{2} \langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) \rangle_0 \times \langle \mathcal{S}_1^2 \rangle_0 . \end{aligned} \quad (3.27)$$

Recalling then formula (3.9) for \mathcal{S}_1 , this correction reads

$$\begin{aligned} \delta_2 \langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) \rangle &= \frac{1}{2(6\pi\alpha')^2} H_{mno} H_{pqr} \int_{\Sigma} d^2 w_1 \int_{\Sigma} d^2 w_2 \\ &\quad \langle :X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2): :X^m(w_1, \bar{w}_1) \partial X^n(w_1) \bar{\partial} X^o(\bar{w}_1): \\ &\quad :X^p(w_2, \bar{w}_2) \partial X^q(w_2) \bar{\partial} X^r(\bar{w}_2): \rangle_0 . \end{aligned} \quad (3.28)$$

The expression in the last two lines can be evaluated using Wick contractions but, as explained in more detail in appendix A.3, the integrals in (3.28) have to be regularized. This can be done by removing a small disc specified by $|w_1 - w_2| < \epsilon$ for $\epsilon \ll 1$ from the integration region, which defines an ultra-violet cutoff. The corresponding computation is shown in appendix A.3 and the result reads

$$\delta_2 \langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) \rangle = \frac{\alpha'^2}{8} H^a_{pq} H^{bpq} \log |z_1 - z_2|^2 \log \epsilon . \quad (3.29)$$

Therefore, the perturbation \mathcal{S}_1 ceases to be marginal at second order in the flux and the theory is not conformally invariant. Writing finally the two-point function

as $\langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) \rangle = -\frac{\alpha'}{2} \log |z_1 - z_2|^2 g^{ab}$, we find a renormalization group flow equation for the inverse world-sheet metric g^{ab} of the form

$$\mu \frac{\partial g^{ab}}{\partial \mu} = -\frac{\alpha'}{4} H^a{}_{pq} H^{bpq} , \quad (3.30)$$

which agrees with equation (3.4) for constant space-time metric, H -flux and dilaton.

3.3 Structure of CFT_H

In equation (3.18) we have seen that the redefined fields $\mathcal{J}^a(z)$ feature a holomorphic three-point function up to first order in the flux. One may therefore suspect that up to linear order in H there exists a conformal field theory, in the following denoted by CFT_H . In this subsection we provide more evidence for this observation by analyzing the operator product expansion (OPE) for the currents and the energy-momentum tensor. Our final goal is to define vertex operators for CFT_H which will allow us to compute string scattering amplitudes.

Current algebra and energy-momentum tensor

Let us first study the fields $\mathcal{J}^a(z)$ and $\bar{\mathcal{J}}^a(\bar{z})$ defined in (3.17) in more detail. Their non-vanishing two-point function up to first order in H is readily found to be

$$\begin{aligned} \langle \mathcal{J}^a(z_1) \mathcal{J}^b(z_2) \rangle &= \langle J^a(z_1) J^b(z_2) \rangle_0 = \frac{\alpha'}{2} \frac{1}{(z_1 - z_2)^2} \delta^{ab} , \\ \langle \bar{\mathcal{J}}^a(\bar{z}_1) \bar{\mathcal{J}}^b(\bar{z}_2) \rangle &= \langle \bar{J}^a(\bar{z}_1) \bar{J}^b(\bar{z}_2) \rangle_0 = \frac{\alpha'}{2} \frac{1}{(\bar{z}_1 - \bar{z}_2)^2} \delta^{ab} , \end{aligned} \quad (3.31)$$

where we employed the definition (3.13) of the currents $J^a(z)$ as well as the two-point function of the fields $X^a(z, \bar{z})$ shown in (3.12). This result reflects that, even though the coordinates X^a are corrected to \mathcal{X}^a , at linear order in the flux the metric is not. Indeed, in the last subsection we have seen that the metric receives correction only at second order in H . Taking then into account the three-point functions (3.18) of the fields $\mathcal{J}^a(z)$ and $\bar{\mathcal{J}}^a(\bar{z})$, with the help of (3.31) we can construct the following OPEs

$$\begin{aligned} \mathcal{J}^a(z_1) \mathcal{J}^b(z_2) &= \frac{\alpha'}{2} \frac{\delta^{ab}}{(z_1 - z_2)^2} - \frac{\alpha'}{4} \frac{i H^ab{}_c}{z_1 - z_2} \mathcal{J}^c(z_2) + \text{reg.} , \\ \bar{\mathcal{J}}^a(\bar{z}_1) \bar{\mathcal{J}}^b(\bar{z}_2) &= \frac{\alpha'}{2} \frac{\delta^{ab}}{(\bar{z}_1 - \bar{z}_2)^2} + \frac{\alpha'}{4} \frac{i H^ab{}_c}{\bar{z}_1 - \bar{z}_2} \bar{\mathcal{J}}^c(\bar{z}_2) + \text{reg.} , \end{aligned} \quad (3.32)$$

where “reg.” stands for regular terms and where the OPEs between $\mathcal{J}^a(z)$ and $\bar{\mathcal{J}}^b(\bar{z})$ are purely regular. Note that (3.32) defines two independent non-abelian

current algebras with structure constants $f^{ab}_c \simeq H^{ab}_c$. The only difference to the usual expressions is an opposite relative sign for H^{ab}_c between the holomorphic and anti-holomorphic parts.

Next, we analyze the energy-momentum tensor. Since the Kalb-Ramond part of the world-sheet action (3.5) is independent of the world-sheet metric, we obtain the result for the free theory which reads $T(z) = \frac{1}{\alpha'} \delta_{ab} : J^a J^b : (z)$ and $\bar{T}(\bar{z}) = \frac{1}{\alpha'} \delta_{ab} : \bar{J}^a \bar{J}^b : (\bar{z})$. However, due to the antisymmetry of H_{abc} , up to linear order in the flux we can write

$$\mathcal{T}(z) = \frac{1}{\alpha'} \delta_{ab} : \mathcal{J}^a \mathcal{J}^b : (z) , \quad \bar{\mathcal{T}}(\bar{z}) = \frac{1}{\alpha'} \delta_{ab} : \bar{\mathcal{J}}^a \bar{\mathcal{J}}^b : (\bar{z}) . \quad (3.33)$$

The antisymmetry of H furthermore implies that the OPEs of two energy-momentum tensors take the form

$$\begin{aligned} \mathcal{T}(z_1) \mathcal{T}(z_2) &= \frac{c/2}{(z_1 - z_2)^4} + \frac{2 \mathcal{T}(z_2)}{(z_1 - z_2)^2} + \frac{\partial \mathcal{T}(z_2)}{z_1 - z_2} + \text{reg.} , \\ \bar{\mathcal{T}}(\bar{z}_1) \bar{\mathcal{T}}(\bar{z}_2) &= \frac{c/2}{(\bar{z}_1 - \bar{z}_2)^4} + \frac{2 \bar{\mathcal{T}}(\bar{z}_2)}{(\bar{z}_1 - \bar{z}_2)^2} + \frac{\partial \bar{\mathcal{T}}(\bar{z}_2)}{\bar{z}_1 - \bar{z}_2} + \text{reg.} , \end{aligned} \quad (3.34)$$

with $\mathcal{T}(z_1) \bar{\mathcal{T}}(\bar{z}_2)$ regular. We therefore find two copies of the Virasoro algebra with the same central charge c as for the free theory. (In the present case of a flat three-dimensional space, this means $c = 3$.) Moreover, using (3.32) and again the antisymmetry of H , one straightforwardly finds

$$\begin{aligned} \mathcal{T}(z_1) \mathcal{J}^a(z_2) &= \frac{\mathcal{J}^a(z_2)}{(z_1 - z_2)^2} + \frac{\partial \mathcal{J}^a(z_2)}{z_1 - z_2} + \text{reg.} , \\ \bar{\mathcal{T}}(\bar{z}_1) \mathcal{J}^a(z_2) &= \text{reg.} , \end{aligned} \quad (3.35)$$

and analogously for the anti-holomorphic parts. The fields $\mathcal{J}^a(z)$ and $\bar{\mathcal{J}}^a(\bar{z})$ are therefore primary fields of conformal dimension $(1, 0)$ and $(0, 1)$ with respect to $\mathcal{T}(z)$ and $\bar{\mathcal{T}}(\bar{z})$; and in CFT_H they are indeed non-abelian currents.

Vertex operator for the tachyon

Let us now define vertex operators. In analogy to the free theory of a closed string without H -flux, which in a compact space can have momentum p_a and winding w^b , we define left- and right-moving momenta $k_{L/R}$ as

$$k_L^a = p^a + \frac{w^a}{\alpha'} , \quad k_R^a = p^a - \frac{w^a}{\alpha'} . \quad (3.36)$$

The vertex operator for the perturbed theory should then be written in the following way

$$\mathcal{V}(z, \bar{z}) = : \exp(i k_L \cdot \mathcal{X}_L + i k_R \cdot \mathcal{X}_R) : , \quad (3.37)$$

where we again employ the short-hand notation $k_L \cdot \mathcal{X}_L = k_{La} \mathcal{X}_L^a$. The left- and right-moving fields $\mathcal{X}_{L/R}^a$ are obtained via integration of the currents, and their OPEs can be computed from (3.32) as

$$\begin{aligned}\mathcal{J}^a(z_1) \mathcal{X}_L^b(z_2) &= -i \frac{\alpha'}{2} \frac{\delta^{ab}}{z_1 - z_2} + \frac{\alpha'}{4} H^{ab}{}_c \mathcal{J}^c(z_2) \log(z_1 - z_2) + \text{reg.} , \\ \overline{\mathcal{J}}^a(\bar{z}_1) \mathcal{X}_R^b(\bar{z}_2) &= -i \frac{\alpha'}{2} \frac{\delta^{ab}}{\bar{z}_1 - \bar{z}_2} - \frac{\alpha'}{4} H^{ab}{}_c \overline{\mathcal{J}}^c(\bar{z}_2) \log(\bar{z}_1 - \bar{z}_2) + \text{reg.} .\end{aligned}\tag{3.38}$$

Now, recall that in the free theory the tachyon vertex operator is a primary field of conformal dimension $(h, \bar{h}) = (\frac{\alpha'}{4} k_L^2, \frac{\alpha'}{4} k_R^2)$, and in covariant quantization of the bosonic string physical states are given by primary fields of conformal dimension $(h, \bar{h}) = (1, 1)$. In the deformed theory CFT_H , we also require that vertex operators $\mathcal{V}(z, \bar{z})$ are primary with respect to $\mathcal{T}(z)$ and $\overline{\mathcal{T}}(\bar{z})$ which, due to the linear corrections (3.38), is not guaranteed a priori. However, it is again the antisymmetry of H which implies

$$\begin{aligned}\mathcal{T}(z_1) \mathcal{V}(z_2, \bar{z}_2) &= \frac{1}{(z_1 - z_2)^2} \frac{\alpha' k_L \cdot k_L}{4} \mathcal{V}(z_2, \bar{z}_2) + \frac{1}{z_1 - z_2} \partial \mathcal{V}(z_2, \bar{z}_2) + \text{reg.} , \\ \overline{\mathcal{T}}(\bar{z}_1) \mathcal{V}(z_2, \bar{z}_2) &= \frac{1}{(\bar{z}_1 - \bar{z}_2)^2} \frac{\alpha' k_R \cdot k_R}{4} \mathcal{V}(z_2, \bar{z}_2) + \frac{1}{\bar{z}_1 - \bar{z}_2} \bar{\partial} \mathcal{V}(z_2, \bar{z}_2) + \text{reg.} .\end{aligned}\tag{3.39}$$

Thus, the vertex operator (3.37) is primary and has conformal dimension $(h, \bar{h}) = (\frac{\alpha'}{4} k_L^2, \frac{\alpha'}{4} k_R^2) = (1, 1)$. It is therefore a physical quantum state of the deformed theory. Classically, this corresponds to the fact that $X_1^a(z, \bar{z})$, as defined in (3.8), solves the classical sigma-model equation of motion (3.6) up to linear order in H , with the zeroth order classical tachyonic solution given by

$$X_0^a(\sigma, \tau) = x^a + \alpha' p^a \tau + w^a \sigma .\tag{3.40}$$

Momentum and winding for the tachyon vertex operator

We continue our discussion and note that in the free theory without flux, the usual vertex operator $V(z, \bar{z}) =: \exp(k_L \cdot X_L + k_R \cdot X_R) :$ carries center of mass momentum p^a and winding ω_a . We want to determine the analogue for the perturbed theory by computing the following OPE in CFT_H

$$\begin{aligned}\mathcal{J}^a(z_1) \mathcal{V}(z_2, \bar{z}_2) &= \frac{1}{z_1 - z_2} \frac{\alpha' k_L^a}{2} \mathcal{V}(z_2, \bar{z}_2) \\ &+ i \frac{\alpha'}{4} \log(z_1 - z_2) H^a{}_{bc} k_L^b : \mathcal{J}^c \mathcal{V} : (z_2, \bar{z}_2) + \text{reg.} ,\end{aligned}\tag{3.41}$$

and similarly for the anti-holomorphic part. Note that in (3.41) a logarithmic term appears, which is also true for more general vertex operators to be discussed below. There are two possibilities to deal with this term:

1. In order to have a well-defined conventional CFT, such terms must be absent implying the constraint $H^a{}_{bc} k_L^b = 0$. The momenta are therefore forced to be transversal to the H -flux which would trivialize most of the results obtained in the following.
2. The second possibility is that generically CFT_H is a logarithmic CFT (LCFT) in which such terms have a physical meaning.

In this paper, we take the latter point of view so that the logarithmic terms should not be eliminated from the very beginning, but should be treated as carrying vanishing conformal dimension. More concretely, we may add a term of the form $\log(\bar{z}_1 - \bar{z}_2) H^a{}_{bc} k_L^b : \mathcal{J}^c \mathcal{V} : (z_2, \bar{z}_2)$ to the OPE (3.41), which is regular in the holomorphic variable z_1 . Therefore, in the above OPE the logarithm can be replaced by $\log|z_1 - z_2|^2$ implying also that (3.41) is single valued.

Let us continue and consider the zero mode \mathcal{P}_L^a of $\mathcal{J}^a(z)$, which can be defined via a contour integral. To determine the \mathcal{P}_L^a eigenvalue of the vertex operator we compute

$$\begin{aligned} \lim_{z_2, \bar{z}_2 \rightarrow 0} \mathcal{P}_L^a \mathcal{V}(z_2, \bar{z}_2) |0\rangle &= \lim_{z_2, \bar{z}_2 \rightarrow 0} \oint \frac{dz_1}{2\pi i} \mathcal{J}^a(z_1) \mathcal{V}(z_2, \bar{z}_2) |0\rangle \\ &= \frac{\alpha' k_L^a}{2} \lim_{z_2, \bar{z}_2 \rightarrow 0} \mathcal{V}(z_2, \bar{z}_2) |0\rangle . \end{aligned} \quad (3.42)$$

Since \mathcal{P}_L^a has a contribution from the Kalb-Ramond part of the sigma-model, it is equivalent to the canonical momentum (though it differs by a numerical prefactor). But, similar to the situation for the energy-momentum tensor, the physical momentum should be related to the uncorrected expression, that is $P_L^a = \oint \frac{dz}{2\pi i} J^a(z)$. Therefore, using (3.17) we compute in the perturbed background (up to first order in H)

$$\begin{aligned} &\lim_{z_2, \bar{z}_2 \rightarrow 0} P_L^a \mathcal{V}(z_2, \bar{z}_2) |0\rangle \\ &= \lim_{z_2, \bar{z}_2 \rightarrow 0} \oint \frac{dz_1}{2\pi i} J^a(z_1) \mathcal{V}(z_2, \bar{z}_2) |0\rangle \\ &= \lim_{z_2, \bar{z}_2 \rightarrow 0} \oint \frac{dz_1}{2\pi i} \left[\mathcal{J}^a(z_1) \mathcal{V}(z_2, \bar{z}_2) + \frac{1}{2} H^a{}_{bc} J^b(z_1) X_R^c(\bar{z}_1) V(z_2, \bar{z}_2) \right] |0\rangle . \end{aligned} \quad (3.43)$$

Note that the first term in the last line is (3.42), and since the second term is already linear in the flux we can work with the free theory. In particular, the second term can be evaluated to be proportional to $H^a{}_{bc} k_L^b k_R^c$, so in order for the tachyon vertex operator in CFT_H to carry momenta (k_L, k_R) we have to require

$$0 = H^a{}_{bc} k_L^b k_R^c \simeq H^a{}_{bc} p^b w^c \simeq [\vec{p} \times \vec{w}]^a , \quad (3.44)$$

where we have used that $H^a{}_{bc} \simeq \epsilon^a{}_{bc}$. Again, this constraint has a corresponding classical analogue. Indeed, taking the classical tachyon (3.40) and requiring

$$\frac{1}{2\pi} \int_0^{2\pi} d\sigma \partial_\tau \mathcal{X}^a = \alpha' p^a, \quad \frac{1}{2\pi} \int_0^{2\pi} d\sigma \partial_\sigma \mathcal{X}^a = \omega^a, \quad (3.45)$$

yields again the constraint (3.44).⁴ Intriguingly, it can also be derived by requiring that the vertex operator of the free-theory $V(z, \bar{z})$ is a primary field of the perturbed one. As can readily be shown, classically this means that the free tachyon solution (3.40) remains a solution of the H -corrected equation of motion when (3.44) is satisfied. We will analyze the consequences of this constraint further in section 3.4.

Vertex operator for the graviton

Let us also consider the vertex operator for the “graviton” in the perturbed theory CFT_H , which we define as

$$\mathcal{V}_G(z, \bar{z}) = \zeta_{ab} : \mathcal{J}^a \bar{\mathcal{J}}^b \exp(ik_L \cdot \mathcal{X}_L + ik_R \cdot \mathcal{X}_R) :. \quad (3.46)$$

The OPE with the holomorphic energy-momentum tensor can be computed using (3.38). Employing the antisymmetry of H_{abc} , one finds

$$\begin{aligned} \mathcal{T}(z_1) \mathcal{V}_G(z_2, \bar{z}_2) &= \frac{\alpha'}{2} \frac{\zeta_{ab} k_L^a}{(z_1 - z_2)^3} : \bar{\mathcal{J}}^b \exp(ik \cdot \mathcal{X}) : \\ &+ \left[\frac{\frac{\alpha'}{4} k_L^2 + 1}{(z_1 - z_2)^2} + \frac{\partial}{z_1 - z_2} \right] \mathcal{V}_G(z_2, \bar{z}_2) \\ &+ \frac{i\alpha'}{4} \frac{1 + \log(z_1 - z_2)}{(z_1 - z_2)^2} H^a{}_{cd} k_L^c \zeta_{ab} : \mathcal{J}^d \bar{\mathcal{J}}^b \exp(ik \cdot \mathcal{X}) :, \end{aligned} \quad (3.47)$$

and similarly for the anti-holomorphic part. Requiring that \mathcal{V}_G is a primary field of conformal dimension $(1, 1)$, the first two lines in (3.47) give the usual on-shell conditions $\zeta_{ab} p^a = \zeta_{ab} \omega^a = 0$ and $k_L^2 = k_R^2 = 0$.

Note that we again obtain a logarithmic term in the OPE, which we could either require to be absent from the very beginning or interpret as an indication for a logarithmic CFT. In the first case, the last line in (3.47) (and its anti-holomorphic counterpart) implies the additional transversality conditions

$$H^{abc} p_b \zeta_{cd} = H^{abc} p_b \zeta_{dc} = 0, \quad H^{abc} w_b \zeta_{cd} = H^{abc} w_b \zeta_{dc} = 0. \quad (3.48)$$

⁴For $\vec{p} \times \vec{w} \neq \vec{0}$ the physical momenta are not conserved, which is the higher-dimensional analogue of the well-known cyclotron orbits arising for point particles moving in a constant magnetic field.

In the second case, tolerating the log-term, we find that the energy-momentum tensor does not act diagonally. For instance, in the case of a three-torus \mathbb{T}^3 one can see that $L_0 = 1$ leads to the mass eigenvalues

$$m_L^2 = 0, \quad m_L^2 = \pm \theta \left(\sum_{a=1}^3 (k_L^a)^2 \right)^{\frac{1}{2}}, \quad (3.49)$$

where θ is given by the flux parameter $\theta_{abc} = \theta \epsilon_{abc}$ and the sum runs only over the momenta longitudinal to the \mathbb{T}^3 . Thus, some of the former massless states become massive and in particular tachyonic, though level matching will eliminate some of these states. Therefore, we obtain the physically acceptable result that some of the longitudinal fluctuations around the constant H -flux background become massive. It would be interesting to completely determine the mass spectrum for this LCFT, but here (in particular in section 4.3) we take a different approach and identify the new mass spectrum of the theory via the poles of the four-tachyon scattering amplitude.

3.4 T-duality

As expected from the string equations of motion, in the last subsection we have found a bona-fide conformal field theory CFT_H , which describes the sigma-model for a flat metric and constant H -flux up to linear order. However, we are also interested in backgrounds T-dual to the H -flux configuration.

On the level of the CFT, T-duality is usually realized as a reflection on the right-moving coordinates. Since the corrected fields $\mathcal{X}^a(z, \bar{z})$ still admit a split into a holomorphic and an anti-holomorphic piece, we define T-duality on the world-sheet action along direction \mathcal{X}^a as

$$\begin{array}{ccc} \mathcal{X}_L^a(z) & \xrightarrow{\text{T-duality}} & +\mathcal{X}_L^a(z), \\ \mathcal{X}_R^a(\bar{z}) & & -\mathcal{X}_R^a(\bar{z}). \end{array} \quad (3.50)$$

Clearly, for the currents this implies

$$\begin{array}{ccc} \mathcal{J}^a(z) & \xrightarrow{\text{T-duality}} & +\mathcal{J}^a(z), \\ \bar{\mathcal{J}}^a(\bar{z}) & & -\bar{\mathcal{J}}^a(\bar{z}), \end{array} \quad (3.51)$$

and so the “structure constants” H^{ab}_c in the the anti-holomorphic OPE (3.32) receive an additional minus sign when performing a T-duality transformation.

In the next section, we compute scattering amplitudes for tachyon vertex operators in the H -flux background. There we allow for both momentum and winding along the directions of our three-dimensional (compact) space specified by (3.3). From table 1 we infer that these scattering amplitudes in the H -flux background are related to the scattering of appropriate momentum and winding

H -flux	ω -flux	Q -flux	R -flux
$\langle p_1, p_2, p_3 \rangle^- \quad \checkmark$	$\langle p_1, p_2, w_3 \rangle^- \quad \checkmark$	$\langle p_1, w_2, w_3 \rangle^- \quad \checkmark$	$\langle w_1, w_2, w_3 \rangle^- \quad \checkmark$
$\langle p_1, p_2, w_3 \rangle^+ \quad \times$	$\langle p_1, p_2, p_3 \rangle^+ \quad \times$	$\langle p_1, w_2, p_3 \rangle^+ \quad \times$	$\langle w_1, w_2, p_3 \rangle^+ \quad \times$
$\langle p_1, w_2, w_3 \rangle^- \quad \times$	$\langle p_1, w_2, p_3 \rangle^- \quad \times$	$\langle p_1, p_2, p_3 \rangle^- \quad \times$	$\langle w_1, p_2, p_3 \rangle^- \quad \times$
$\langle w_1, w_2, w_3 \rangle^+ \quad \checkmark$	$\langle w_1, w_2, p_3 \rangle^+ \quad \checkmark$	$\langle w_1, p_2, p_3 \rangle^+ \quad \checkmark$	$\langle p_1, p_2, p_3 \rangle^+ \quad \checkmark$

Table 2: T-duality relations of momentum and winding mode scattering in the four three-form flux backgrounds. Entries in the same row are related via the T-dualities from table 1, and the upper index indicates the relative sign between the holomorphic and anti-holomorphic part in the three-point function $\langle \mathcal{X}^a \mathcal{X}^b \mathcal{X}^c \rangle$. The symbol thereafter indicates whether the condition (3.44) is satisfied.

states in the ω -, Q - and R -flux backgrounds. However, in the T-dual models, we are particularly interested in pure momentum scattering, as from there one would derive the low-energy effective action as a (ordinary) derivative expansion. Now, in the previous section we have seen that a tachyon vertex operator $\mathcal{V}(z, \bar{z})$ indeed corresponds to a physical state, but its usual momentum and winding quantum numbers are only uncorrected if $\vec{p} \times \vec{w} = \vec{0}$. Recall that classically this means that the free tachyon solution of the sigma-model equations of motion remains to be a solution of the H -corrected ones. Thus, we expect that after imposing the above constraint, any effect we derive at linear order in H cannot be a consequence of the linear redefinition of the classical solution of the tachyon, but reflects a property of the uncorrected solution.

In table 2 we have made the T-duality relations more explicit, which we explain in some detail.

- From the first row we infer that the effective field theory (with space-time derivatives $\frac{\partial}{\partial X^a}$) for tachyons in the H -flux background is expected to be reliably computable (in the sense explained above) from scattering amplitudes of pure momentum tachyons, since in this case (3.44) is satisfied. The basic three-point function of the coordinates is then given by (3.26). Therefore, from now on we denote

$$\langle \mathcal{X}^a(z_1, \bar{z}_1) \mathcal{X}^b(z_2, \bar{z}_2) \mathcal{X}^c(z_3, \bar{z}_3) \rangle^- \stackrel{\text{def}}{=} \theta^{abc} \left[\mathcal{L}\left(\frac{z_{12}}{z_{13}}\right) - \mathcal{L}\left(\frac{\bar{z}_{12}}{\bar{z}_{13}}\right) \right]. \quad (3.52)$$

- The next row shows that a scattering of pure momentum modes in the geometric flux background is related to the scattering of (p_1, p_2, w_3) modes in the H -flux background by T-duality. However, in this case $\vec{w} \times \vec{p} \neq \vec{0}$ and we cannot exclude that any effect we derive at linear order in H is just reflecting the linear redefinition (3.8) of the space coordinate. The same situation occurs for pure momentum scattering in the Q -flux background.

Flux	Commutators	Three-brackets
H -flux	$[\tilde{X}^2, \tilde{X}^3] \simeq w_1$	$[\tilde{X}^2, \tilde{X}^3, \tilde{X}^1]$
ω -flux	$[\tilde{X}^2, X^3] \simeq w_1$	$[\tilde{X}^2, X^3, \tilde{X}^1]$
Q -flux	$[X^2, X^3] \simeq w_1$	$[X^2, X^3, \tilde{X}^1]$
R -flux	$[X^2, X^3] \simeq p_1$	$[X^2, X^3, X^1]$

Table 3: Non-vanishing commutators and three-brackets in the four flux backgrounds.

- The last row in the table shows that only for the case of R -flux we can again reliably compute the scattering amplitudes for pure momentum tachyons. By T-duality, they are related to the scattering of pure winding states in the H -flux background. Employing T-duality for this R -flux background, the basic three-point function for pure momentum states reads

$$\langle \mathcal{X}^a(z_1, \bar{z}_1) \mathcal{X}^b(z_2, \bar{z}_2) \mathcal{X}^c(z_3, \bar{z}_3) \rangle^+ \stackrel{\text{def}}{=} \theta^{abc} \left[\mathcal{L}\left(\frac{z_{12}}{z_{13}}\right) + \mathcal{L}\left(\frac{\bar{z}_{12}}{\bar{z}_{13}}\right) \right]. \quad (3.53)$$

These considerations are also consistent with the results in [15] from which the commutation relations between coordinates X^a and their duals \tilde{X}^a can be obtained.⁵ Furthermore, using the canonical commutation relations $[X^1, p_1] = \text{const.}$ and $[\tilde{X}^1, w_1] = \text{const.}$, one can derive three-brackets for the four different cases. These results are summarized in table 3.

4 Tachyon scattering amplitudes

In this section, by computing higher N -point scattering amplitudes of tachyon vertex operators and discussing their pole structure, we want to infer properties of the theory. This is in the same spirit as for the famous Veneziano and Virasoro-Shapiro four-point amplitudes, which contain information about the Regge resonances as well as about the three-point couplings involving two tachyons.

4.1 Three-tachyon amplitude

We start with the three-tachyon amplitude. As discussed above, we focus on a compact three-dimensional space and therein we are interested in pure momentum (p_1, p_2, p_3) or pure winding (w_1, w_2, w_3) state scattering, where the latter is related by three T-dualities to pure momentum scattering in the R -flux background. We

⁵More precisely, the ω -, Q - and R -flux cases with elliptic monodromies were explicitly discussed in [15]; the commutation relations for parabolic fluxes will be discussed in [28].

therefore consider vertex operators of form

$$\begin{aligned}\mathcal{V}_i^- &\equiv \mathcal{V}_{p_i}(z_i, \bar{z}_i) = : \exp(ip_i \cdot \mathcal{X}(z_i, \bar{z}_i)) : , \\ \mathcal{V}_i^+ &\equiv \mathcal{V}_{w_i}(z_i, \bar{z}_i) = : \exp(iw_i \cdot \tilde{\mathcal{X}}(z_i, \bar{z}_i)) : ,\end{aligned}\tag{4.1}$$

where $\tilde{\mathcal{X}} = \mathcal{X}_L - \mathcal{X}_R$. Note that here and in the following we employ the short hand notation \mathcal{V}_i^\mp and V_i^\mp for the vertex operators of the perturbed and free theory, respectively. Furthermore, since we can consider \mathcal{V}_i^+ as the momentum vertex operator in the R -flux background, in the following we set $w|_H \rightarrow p|_R$. However, in string theory one has to work with vertex operators integrated over the world-sheet. We therefore define

$$\mathcal{T}_i^\mp = \int d^2z \mathcal{V}_i^\mp .\tag{4.2}$$

Taking into account the freedom to fix three points on the world-sheet via the $SL(2, \mathbb{C})$ symmetry, the three-tachyon scattering amplitude is given by

$$\langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \rangle^\mp = \int \prod_{i=1}^3 d^2z_i \delta^{(2)}(z_i - z_i^0) |z_{12} z_{13} z_{23}|^2 \langle \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \rangle^\mp ,\tag{4.3}$$

where we have put the superscript \mp indicating the H -flux and R -flux background outside the bracket in order to shorten the notation. Using the general formula (A.55) given in appendix A.3, we obtain for correlator of three vertex operators

$$\langle \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \rangle^\mp = \frac{\delta(p_1 + p_2 + p_3)}{|z_{12} z_{13} z_{23}|^2} \exp \left[-i\theta^{abc} p_{1,a} p_{2,b} p_{3,c} \left[\mathcal{L}\left(\frac{z_{12}}{z_{13}}\right) \mp \mathcal{L}\left(\frac{\bar{z}_{12}}{\bar{z}_{13}}\right) \right] \right]_\theta ,\tag{4.4}$$

where $[\dots]_\theta$ indicates that the result is valid only up to linear order in θ . The full scattering amplitude then becomes

$$\begin{aligned}\langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \rangle^\mp &= \int \prod_{i=1}^3 d^2z_i \delta^{(2)}(z_i - z_i^0) \delta(p_1 + p_2 + p_3) \times \\ &\exp \left[-i\theta^{abc} p_{1,a} p_{2,b} p_{3,c} \left[\mathcal{L}\left(\frac{z_{12}}{z_{13}}\right) \mp \mathcal{L}\left(\frac{\bar{z}_{12}}{\bar{z}_{13}}\right) \right] \right]_\theta .\end{aligned}\tag{4.5}$$

Let us now study the behavior of (4.4) under permutations of the vertex operators \mathcal{V}_i^\mp . Before applying momentum conservation, the three-tachyon amplitude for a permutation σ of the vertex operators can be computed using the relations (3.25). With $\epsilon = -1$ for the H -flux and $\epsilon = +1$ for the R -flux, one finds⁶

$$\langle \mathcal{V}_{\sigma(1)} \mathcal{V}_{\sigma(2)} \mathcal{V}_{\sigma(3)} \rangle^\epsilon = \exp \left[i \left(\frac{1+\epsilon}{2} \right) \eta_\sigma \pi^2 \theta^{abc} p_{1,a} p_{2,b} p_{3,c} \right] \langle \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \rangle^\epsilon ,\tag{4.6}$$

⁶Exploiting the freedom of adding integration constants to (3.22), we could have chosen $-\frac{3L(1)}{2} + \mathcal{L}\left(\frac{z_{12}}{z_{13}}\right)$ for the basic holomorphic three-point function. Then, the phase in (4.6) would be vanishing also in the case of R -flux. However, such a constant term corresponds to a constant shift in the Jacobi-identity (3.1), implying it to be generically non-vanishing, which we consider to be unnatural. Therefore, comparing with the analogous computation for the open string [29], we choose the integration constant to be zero in the present situation.

where in addition $\eta_\sigma = 1$ for an odd permutation and $\eta_\sigma = 0$ for an even one. Thus, for the R -flux background a non-trivial phase may appear which, in this paper, we have established up to linear order in the flux. This situation is similar to the open string where an analogous phase hinted towards a noncommutative star-product (see section 2.1). Indeed, as will be discussed in more detail in section 5.2, the phase in (4.6) can be recovered from a three-product on the space of functions $V_{p_n}(x) = \exp(i p_n \cdot x)$, which can be defined as

$$V_{p_1}(x) \triangle V_{p_2}(x) \triangle V_{p_3}(x) \stackrel{\text{def}}{=} \exp\left(-i \frac{\pi^2}{2} \theta^{abc} p_{1,a} p_{2,b} p_{3,c}\right) V_{p_1+p_2+p_3}(x) . \quad (4.7)$$

However, in correlation functions operators are understood to be radially ordered and so changing the order of operators should not change the form of the amplitude. This is known as crossing symmetry which is one of the defining properties of a CFT and thus should also be satisfied for our CFT_H . In the case of the R -flux background, this is reconciled by applying momentum conservation leading to

$$p_{1,a} p_{2,b} p_{3,c} \theta^{abc} = 0 \quad \text{for} \quad p_3 = -p_1 - p_2 . \quad (4.8)$$

Therefore, scattering amplitudes of three tachyons do not receive any corrections at linear order in θ both for the H - and R -flux. We therefore find

$$\langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \rangle^\mp = \delta(p_1 + p_2 + p_3) . \quad (4.9)$$

This is analogous to the situation in noncommutative open string theory, where the two-point function (2.4) does not receive any corrections.

4.2 N-tachyon amplitudes

In analogy to the result for the open string shown in equation (2.4), we now want to detect phases possibly appearing for the product of N closed string tachyon vertex operators. Before we consider the general case, let us start with the amplitude of four tachyons. Employing the general formula (A.55), up to linear order in θ we obtain

$$\begin{aligned} \langle \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \mathcal{V}_4 \rangle^\mp &= \langle V_1 V_2 V_3 V_4 \rangle_0^\mp \times \\ &\exp \left[-i \theta^{abc} \sum_{1 \leq i < j < k \leq 4} p_{i,a} p_{j,b} p_{k,c} \left[\mathcal{L}\left(\frac{z_{ij}}{z_{ik}}\right) \mp \mathcal{L}\left(\frac{\bar{z}_{ij}}{\bar{z}_{ik}}\right) \right] \right]_\theta . \end{aligned} \quad (4.10)$$

Again, the difference between H - and R -flux is given by the sign between the holomorphic and the anti-holomorphic contribution, and the four-point function $\langle V_1 V_2 V_3 V_4 \rangle_0^\mp$ is just the one from the free theory. We can now determine the behavior of the amplitude under a permutation of the vertex operators. Prior to

using momentum conservation, invoking the fundamental identity of the Rogers dilogarithm (3.25), we again find momentum dependent phase factors. Analogous to the three-tachyon amplitude, these arise in the case of R -flux and can be described as resulting from a deformed four-product of the form

$$V_{p_1}(x) \Delta_4 V_{p_2}(x) \Delta_4 V_{p_3}(x) \Delta_4 V_{p_4}(x) \stackrel{\text{def}}{=} \exp \left[-i \frac{\pi^2}{2} \theta^{abc} (p_{1,a} p_{2,b} p_{3,c} + p_{1,a} p_{2,b} p_{4,c} + p_{1,a} p_{3,b} p_{4,c} + p_{2,a} p_{3,b} p_{4,c}) \right] V_{\sum p_i}(x). \quad (4.11)$$

However, employing momentum conservation, one can show that this phase becomes trivial so that the four-tachyon amplitude is indeed crossing symmetric.

This computation for four tachyons can straightforwardly be generalized to higher N -tachyon amplitudes for which we find

$$\langle \mathcal{V}_1 \mathcal{V}_2 \dots \mathcal{V}_N \rangle^\mp = \langle V_1 V_2 \dots V_N \rangle_0^\mp \times \exp \left[-i \theta^{abc} \sum_{1 \leq i < j < k \leq N} p_{i,a} p_{j,b} p_{k,c} \left[\mathcal{L} \left(\frac{z_{ij}}{z_{ik}} \right) \mp \mathcal{L} \left(\frac{\bar{z}_{ij}}{\bar{z}_{ik}} \right) \right] \right]_\theta. \quad (4.12)$$

The phase factors appearing when permuting two vertex operators for the case of the R -flux background can be encoded in a deformed N -product of the form

$$V_{p_1}(x) \Delta_N \dots \Delta_N V_{p_N}(x) \stackrel{\text{def}}{=} \exp \left(-i \frac{\pi^2}{2} \theta^{abc} \sum_{1 \leq i < j < k \leq N} p_{i,a} p_{j,b} p_{k,c} \right) V_{\sum p_i}(x). \quad (4.13)$$

The phase becomes again trivial after employing momentum conservation so that all N -tachyon correlators are crossing symmetric. This signals that the basic principle of perturbative closed string theory, namely conformal field theory, seems to be compatible with non-geometric backgrounds for which the N -product of functions is deformed by (4.13).

4.3 The fluxed Virasoro-Shapiro amplitude

The four-tachyon scattering amplitude has played an important role in the history of string theory. In fact, many aspects of the theory were detected by analyzing its properties. In a similar spirit, we now further elaborate on the four-tachyon scattering amplitude (4.10). Using momentum conservation $\sum_i p_i = 0$ to eliminate p_4 , one obtains

$$\langle \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \mathcal{V}_4 \rangle^\mp = \langle V_1 V_2 V_3 V_4 \rangle_0^\mp \times \exp \left[-i \theta^{abc} p_{1,a} p_{2,b} p_{3,c} \left[\mathcal{L} \left(\frac{z_{12}}{z_{13}} \right) - \mathcal{L} \left(\frac{z_{12}}{z_{14}} \right) + \mathcal{L} \left(\frac{z_{13}}{z_{14}} \right) - \mathcal{L} \left(\frac{z_{23}}{z_{24}} \right) \mp \text{c.c.} \right] \right]_\theta. \quad (4.14)$$

Next, we simplify the sum over the four \mathcal{L} -functions by using the five-term relation of the (complex) Rogers dilogarithm. Unfortunately, as indicated in appendix

A.2, this relation becomes quite involved in the complex case since logarithmic corrections of the form $F(z_{ij}) = \sum \log z_{ij}$ appear. However, observing that they satisfy $\partial_i \partial_j \partial_k F(z_{mn}) = 0$ for all $i, j, k \in \{1, 2, 3, 4\}$ and $i \neq j \neq k \neq i$, they appear not to be physical. These corrections are closely related to the existence of branch cuts and are very effectively described by the so-called *extended Rogers dilogarithm*. As shown in detail in appendix A.2, this allows us to express the four-point function (4.14) in terms of $\mathcal{L}(X)$, where the cross-ratio X is defined as

$$X = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}. \quad (4.15)$$

As it is also shown in appendix A.2, requiring that the four-tachyon correlator is crossing symmetric leads to the $SL(2, \mathbb{C})$ invariant and explicitly crossing symmetric four-tachyon amplitude

$$\langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \rangle^\mp = \int d^2 X \frac{\exp \left[-i\theta^{abc} p_a^1 p_b^2 p_c^3 \left[\left(-\frac{3}{2}L(1) + \mathcal{L}(X) \right) \mp \left(-\frac{3}{2}L(1) + \mathcal{L}(\overline{X}) \right) \right] \right]_\theta}{|X|^{2-2a} |1-X|^{2-2c}}, \quad (4.16)$$

where the integrated tachyon vertex operator \mathcal{T}_i^\mp was defined in (4.2) and where we employed

$$a = \frac{\alpha'}{4}(p_1 + p_4)^2 - 1, \quad b = \frac{\alpha'}{4}(p_1 + p_3)^2 - 1, \quad c = \frac{\alpha'}{4}(p_1 + p_2)^2 - 1. \quad (4.17)$$

The three corresponding Mandelstam variables read $u = -(p_1 + p_4)^2$, $t = -(p_1 + p_3)^2$ and $s = -(p_1 + p_2)^2$, and the on-shell external tachyons satisfy $\alpha' p_i^2 = 4$ so that $a + b + c = 1$. We consider (4.16) as the generalization of the Virasoro-Shapiro amplitude, which in our case includes corrections up to first order in θ . We therefore call it the (linearized) fluxed Virasoro-Shapiro (FVS) amplitude.

The usual Virasoro-Shapiro amplitude is just its $\theta = 0$ limit, in which case the integral can be solved in closed form. The result reads

$$\langle T_1 T_2 T_3 T_4 \rangle_0 = 2\pi \frac{\Gamma(a) \Gamma(b) \Gamma(c)}{\Gamma(a+b) \Gamma(a+c) \Gamma(b+c)}, \quad (4.18)$$

which, as it is well-known, has single poles at $a, b, c = -n$ for $n \in \mathbb{Z}_0^+$ corresponding to the Regge excitations of mass $m_n^2 = \frac{4}{\alpha'}(n-1)$. Furthermore, unitarity implies that on a resonance R_n of mass m , the four-tachyon amplitude factorizes into two three-point functions

$$\langle T_1 T_2 T_3 T_4 \rangle_0 \stackrel{u \rightarrow -m^2}{\simeq} \frac{\langle T(p_1) T(p_4) R_n(-p_1 - p_4) \rangle_0 \langle T(p_2) T(p_3) R_n(-p_2 - p_3) \rangle_0}{(p_1 + p_4)^2 + m_n^2}. \quad (4.19)$$

This factorization does not only occur in the u -channel, as presented, but similarly also in the s - and t -channel.

In the same way as the Virasoro-Shapiro amplitude contains information about the underlying theory, we expect that also (4.16) does contain new information on the scattering of four tachyons in a small three-form flux background. Ideally, one would solve the integral in a closed form, for instance by the method of Kawai-Lewellen-Tye (KLT) [30]. However, even without having such an explicit result at our disposal, we can proceed and analyze the pole structure of the amplitude. Thus, let us consider the correction linear in the flux which reads

$$\begin{aligned} & \delta_1 \langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \rangle^\mp \\ &= -i\theta^{abc} p_{1,a} p_{2,b} p_{3,c} \int d^2 X \frac{[-\frac{3}{2}L(1) + \mathcal{L}(X)] \mp [-\frac{3}{2}L(1) + \mathcal{L}(\bar{X})]}{|X|^{2-2a} |1-X|^{2-2c}}. \end{aligned} \quad (4.20)$$

We are interested in corrections to the (former) resonances at $u = \frac{4}{\alpha'}(n-1)$. These divergences originate from the region $|X| < 1$ and can be computed by first introducing polar coordinates (r, φ) and then expanding the integrand as a series in r around the origin. In this respect it is useful to invoke the relation (see the end of appendix A.2)

$$\mathcal{L}(x) = 3L(x) \mp \frac{i\pi}{2} \log(x(1-x)) \ , \quad x \in \mathbb{C} \ , \quad (4.21)$$

with the minus sign corresponding to the upper half plane, $\text{Im}(x) > 0$, and the plus sign to the lower half plane $\text{Im}(x) < 0$. Moreover, besides the basic relation $\log X = \log r + i\varphi$, it is easy to verify the following expansions

$$\begin{aligned} \text{Re}(\log(1-X)) &= -\sum_{n=1}^{\infty} \frac{r^n}{n} \cos(n\varphi) \ , & \text{Im}(\log(1-X)) &= -\sum_{n=1}^{\infty} \frac{r^n}{n} \sin(n\varphi) \ , \\ \text{Re}(\text{Li}_2(X)) &= \sum_{n=1}^{\infty} \frac{r^n}{n^2} \cos(n\varphi) \ , & \text{Im}(\text{Li}_2(X)) &= \sum_{n=1}^{\infty} \frac{r^n}{n^2} \sin(n\varphi) \ . \end{aligned} \quad (4.22)$$

It remains to expand the $1/|1-X|^{2-2c}$ term, for which we find

$$\begin{aligned} \frac{1}{|1-X|^{2-2c}} &= \sum_{n=0}^{\infty} a_n(\varphi) r^n \\ &= 1 + [-2(c-1) \cos(\varphi)] r + [(c-1) + 2(c-1)(c-2) \cos^2(\varphi)] r^2 \\ &\quad + [-2(c-1)(c-2) \cos(\varphi) - \frac{4}{3}(c-1)(c-2)(c-3) \cos^3(\varphi)] r^3 \\ &\quad + O(r^4) \ . \end{aligned} \quad (4.23)$$

Pole structure for H -flux

With the help of the above expressions, we can now analyze the pole structure of the FVS-amplitude at linear order in θ^{abc} . We start with the case of H -flux which corresponds to the minus sign in (4.20). The r -expansion at linear order in θ reads

$$\begin{aligned} \delta_1 \langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \rangle^- &= 2\theta^{abc} p_{1,a} p_{2,b} p_{3,c} \int_0^1 dr \int_{-\pi}^{+\pi} d\varphi r^{2a-1} \sum_{n=0}^{\infty} a_n(\varphi) r^n \times \\ &\left[3 \sum_{n=1}^{\infty} \frac{r^n}{n^2} \sin(n\varphi) - \frac{3}{2} \log(r) \sum_{n=1}^{\infty} \frac{r^n}{n} \sin(n\varphi) \right. \\ &\left. - \frac{3}{2} \varphi \sum_{n=1}^{\infty} \frac{r^n}{n} \cos(n\varphi) \mp \frac{\pi}{2} \log(r) \pm \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{r^n}{n} \cos(n\varphi) \right], \end{aligned} \quad (4.24)$$

where again the upper sign holds in the upper half-plane $0 \leq \varphi \leq \pi$ while the lower sign indicates the lower half-plane $-\pi \leq \varphi \leq 0$. We have furthermore restricted the integration to a disk of unit radius around the origin in the complex plane. This allows us to read off the poles in $a = \frac{\alpha'}{4}(p_1 + p_4)^2 - 1$ since $\int_0^1 dr r^{2a+n-1} = (2a+n)^{-1}$ for $n \geq 0$.

At each order in r , the above integral trivially vanishes since the integrand is anti-symmetric in φ . Thus, at linear order in H there are no corrections to the exchange modes at a particular resonance R_n . For the tachyon exchange, this is consistent with what we found from the three-point amplitudes in section 4.1. We will discuss the origin for the absence of corrections at the end of this section.

Pole structure for R -flux

We now perform the same computation for the case of non-geometric R -flux, in which case the lower sign in (4.20) applies. We then find the following expansion

$$\begin{aligned} \delta_1 \langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \rangle^+ &= -2i\theta^{abc} p_{1,a} p_{2,b} p_{3,c} \int_0^1 dr \int_{-\pi}^{+\pi} d\varphi r^{2a-1} \sum_{n=0}^{\infty} a_n(\varphi) r^n \times \\ &\left[-\frac{3}{2} L(1) + 3 \sum_{n=1}^{\infty} \frac{r^n}{n^2} \cos(n\varphi) - \frac{3}{2} \log(r) \sum_{n=1}^{\infty} \frac{r^n}{n} \cos(n\varphi) \right. \\ &\left. + \frac{3}{2} \varphi \sum_{n=1}^{\infty} \frac{r^n}{n} \sin(n\varphi) \pm \frac{\pi}{2} \varphi \mp \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{r^n}{n} \sin(n\varphi) \right]. \end{aligned} \quad (4.25)$$

In this situation, at linear order in the flux there are non-vanishing contributions. Let us discuss the first three leading terms in some more detail:

a) The tachyon pole

The first potential pole appears at order r^{2a-1+n} for $n = 0$, which after integration gives $(2a)^{-1}$ and thus corresponds to tachyon $(p_1 + p_4)^2 = \frac{4}{\alpha'}$. The φ integral in this case is evaluated as

$$\int_{-\pi}^{\pi} d\varphi \left(-\frac{3}{2}L(1) \pm \frac{\pi}{2}\varphi \right) = 2 \int_0^{\pi} d\varphi \left(-\frac{\pi^2}{4} + \frac{\pi}{2}\varphi \right) = 0. \quad (4.26)$$

Therefore, by factorization the three-tachyon amplitude must vanish at linear order in θ , which is consistent with the result obtained by direct computation in the previous section, and with the implicit assumption to put the external tachyons on the mass-shell $m^2 = -\frac{4}{\alpha'}$.

b) A new tachyonic pole

At order r^{2a-1+n} with $n = 1$ the integration gives $(2a + 1)^{-1}$ corresponding to a resonance which, due to level matching, was not part of the physical spectrum of the 26-dimensional free bosonic string. After some steps of computation we obtain

$$\begin{aligned} \delta_1 \langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \rangle^+ &\stackrel{u \rightarrow -\frac{2}{\alpha'}}{\simeq} -2i\theta^{abc} p_{1,a} p_{2,b} p_{3,c} \pi \frac{b-c}{2a+1} \\ &\simeq -2i\theta^{abc} p_{1,a} p_{2,b} p_{3,c} \pi \frac{p_{14} \cdot p_{23}}{(p_1 + p_4)^2 - \frac{2}{\alpha'}}, \end{aligned} \quad (4.27)$$

where we note that a, b, c were defined in (4.17) and where we employ $p_{ij} = p_i - p_j$. As a qualitatively new feature, the mass shift is *not continuous* because, once we turn on θ even only infinitesimally, the new mass level appears. Via level matching, this mode appears to be a \mathbb{Z}_2 twisted mass state. It would be very interesting to understand the origin of this mode in more detail. We may speculate that a former graviton mode becomes tachyonic or that unphysical modes becomes physical due to the flux. In any case, this new tachyon signals an instability of the system, which we will discuss in more detail in section 5.1. In the following, we call this new tachyon a tachyon of type I.

c) The graviton pole

At order r^{2a-1+n} for $n = 2$ the exchange particle is the graviton. One obtains various contributions, but the qualitatively new feature is an integral involving a $\log(r)$ term. Anticipating the result for higher orders in θ , one is led to integrals involving $(\log r)^m$ with $m > 1$.⁷ Integrals of this type can be computed explicitly as follows

$$\int_0^1 dr r^{2a+n-1} (\log r)^m = \frac{(-1)^m m!}{(2a+n)^{m+1}}, \quad (4.28)$$

⁷Here we are assuming that the corrections to tachyon correlators at higher order in the flux indeed give the exponential shown in (4.16) beyond linear order.

and lead to higher order poles in the Mandelstam variables. This seems to be in contradiction with the interpretation in terms of Regge resonances. However, a mass renormalization of such a Regge resonance would indeed induce higher order poles of the form

$$\frac{1}{p^2 + m^2 + (\Delta m)^2} = \frac{1}{p^2 + m^2} - \frac{(\Delta m)^2}{(p^2 + m^2)^2} + \dots \quad (4.29)$$

The total contribution from the graviton exchange can then be expressed as

$$\begin{aligned} \delta_1 \langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \rangle^+ &\stackrel{u \rightarrow 0}{\simeq} i\theta^{abc} p_{1,a} p_{2,b} p_{3,c} \left(\frac{9\pi(b-c)}{8(a+1)} + \frac{3\pi(b-c)}{8(a+1)^2} \right) \\ &\simeq i\theta^{abc} p_{1,a} p_{2,b} p_{3,c} \left(\frac{\pi p_{14} \cdot p_{23}}{(p_1 + p_4)^2} + \frac{3\pi p_{14} \cdot p_{23}}{(p_1 + p_4)^4} \right). \end{aligned} \quad (4.30)$$

We observe that the first simple pole has the same momentum dependence as the new tachyonic mode in (4.27). Due to the mass shift, also these modes can become tachyonic, where in contrast to the tachyon discussed in the last paragraph here the mass-shift *is continuous* in the parameter θ . Furthermore, we recall the computation of L_0 eigenvalues (3.49) for the graviton vertex operators which are in accordance with the fact that a small R -flux induces a light tachyonic mode. In the following such tachyons are called of type II.

d) Higher order poles

Finally, the pole structure for higher Regge excitations is very similar. At each former pole at $a = -n$ one now also finds a double pole and there appears a new pole at $a = -n + \frac{1}{2}$. The general structure can be expressed as

$$\begin{aligned} \delta_1 \langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \rangle^+ &\simeq i\theta^{abc} p_{1,a} p_{2,b} p_{3,c} \sum_{n=1}^{\infty} \left(\frac{(b-c) P_1^{2n-2}(b, c)}{(a+n-\frac{1}{2})} + \right. \\ &\quad \left. \frac{(b-c) P_2^{2n-2}(b, c)}{(a+n)} + \frac{(b-c) P_3^{2n-2}(b, c)}{(a+n)^2} \right), \end{aligned} \quad (4.31)$$

where $P_i^{2n-2}(b, c)$, $i = 1, 2, 3$ are polynomials of order $2n-2$ in b and c . Thus, we see that the appearance of the aforementioned new \mathbb{Z}_2 twisted poles is generic.

Discussion of the pole structure

Let us discuss the form of the poles obtained in b) and c) for the R -flux case in more detail, where part of it will be at a rather qualitative level and might turn out to be too naive.

First, we observe that the numerators for the single and the double pole at linear order in θ have the same external momentum dependence. This provides

reason to believe that all these terms are related to massless gravitons G , dilaton D and Kalb-Ramond fields B at zeroth order in the flux. Let us try to understand which corrections to the two-tachyon-one graviton three point function $\langle \mathcal{T}\mathcal{T}\mathcal{G} \rangle$ can induce these poles in the factorization limit. At zero order in θ , that is for the free theory, it is known that $\langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{G}_3 \rangle^\mp \simeq g_{ab} p_{21}^a p_{21}^b$, i.e. there is only a non-vanishing contribution for gravitons and for the dilaton. Now let us assume that at linear order there is a correction of the form

$$\langle \mathcal{T}(p_1) \mathcal{T}(p_2) \mathcal{G}(p_3) \rangle^\pm \simeq g_{ab} p_{21}^a p_{21}^b + p_{21}^a \theta^{bcd} p_{1,c} p_{2,d} \begin{cases} g_{ab} & R\text{-flux} , \\ b_{ab} & H\text{-flux} , \end{cases} \quad (4.32)$$

and discuss its consequences for the four-tachyon amplitude.

- The two cases in (4.32) imply that at linear order in θ , for H -flux the $\langle TTB \rangle$ three-vertex is corrected and for R -flux the $\langle TTG \rangle$ and $\langle TTD \rangle$ vertices. Thus, in the factorization limit of the four-tachyon amplitude, up to linear order in θ we find

$$\langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \rangle^\mp \xrightarrow{u \rightarrow 0} \begin{cases} \frac{[p_{14} \cdot p_{23}]^2}{\alpha' u} + \frac{(\theta^{abc} p_{1,a} p_{2,b} p_{3,c}) (p_{14} \cdot p_{23})}{\alpha' u} & R\text{-flux} , \\ \frac{[p_{14} \cdot p_{23}]^2}{\alpha' u} & H\text{-flux} , \end{cases} \quad (4.33)$$

where $u = -(p_1 + p_4)^2$ is a Mandelstam variable. In the case of non-geometric R -flux, we find the single graviton pole shown in (4.30), while for H -flux we do not find any linear correction to the graviton pole consistent with (4.3). It is due this property that we included the B -field in (4.32) in the first place so that the expression becomes more symmetric. We will see in a moment that, in view of T-duality, it makes sense that the rôle of zero order graviton and Kalb-Ramond field fluctuations are exchanged by switching from R -flux to H -flux.

- As said, we expect the double pole in (4.30) to arise from a mass shift $\Delta m^2(g_{ab})$ of some of the longitudinal gravitons/dilaton at linear order in θ which corresponds to the diagram shown in figure 1. Since the factorization limit of the four-tachyon amplitude involves a sum over all polarizations

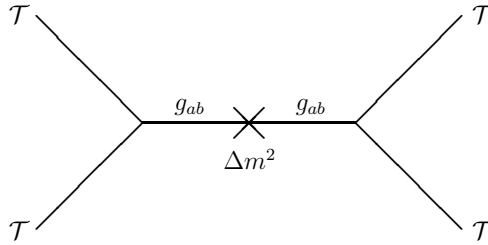


Figure 1: Mass correction of the graviton in the $\langle \mathcal{T}_1 \mathcal{T}_2 \mathcal{T}_3 \mathcal{T}_4 \rangle$ amplitude.

of the graviton, it is not obvious what the separate eigenvalues $\Delta m^2(g_{ab})$ are.⁸ But, interpreting the R -flux as a nonassociative deformation of ordinary space, one could imagine that part of the former general covariance symmetry is broken and the corresponding graviton mode becomes massive. For the case of H -flux there are no double poles but, as before, mass shifts of the longitudinal B_{ab} -fields can remain undetected. In fact, due to breaking of conformal symmetry at second order in H , we expect that also some of the B -field modes become massive in the linear H -flux background. Thus, as mentioned before, under T-duality the rôle of graviton and B -field fluctuations seem to get exchanged.

- Besides these mass shifts linear in θ , the R -flux amplitude has also shown a new tachyon of type I with half the (negative) mass squared of the bosonic tachyonic ground state. For the H -flux case, again we do not directly see such a mode, but via T-duality expect that there also exists a similar tachyonic mode.

To summarize, from the factorization of the four-tachyon amplitudes in the R -flux and H -flux background as well as from T-duality, we infer that in both cases there are two new types of tachyonic modes, which we have denoted by type I and type II. For the case of R -flux, these are expected to be related to zero order graviton/dilaton modes, whereas for H -flux we expect them to be zero order B -field modes.

5 Asymmetric backgrounds and nonassociative geometry

After having gained some insight on the structure of scattering amplitudes, we now provide a physical interpretation of some of the results. First, we explain the appearance of the new tachyons of type I and type II in the spectrum and discuss tachyon condensation. Our line of reasoning will lead to the conjecture that non-geometric R -flux is related to an asymmetric version of the CFT defined via the representation theory of the Kac-Moody algebra $\widehat{su}(2)_k$, i.e. the CFT of the WZW model.

Second, we give arguments for a new space-time structure responsible for the appearance of constant phase factors in the N -tachyon amplitudes (prior to invoking momentum conservation). More concretely, as already indicated in the previous section, we define a new tri-product inducing a product of N functions on a nonassociative space, which makes the proposal that non-geometric R -flux is related to nonassociative geometry more precise.

⁸A more detailed analysis of the off-diagonal terms in the logarithmic OPE (3.47) might allow to determine the precise eigenvalues.

5.1 Speculations about tachyon condensation

From our analysis of the the four-tachyon amplitude we concluded that there appear two types of new tachyonic modes. They are different in the sense that for non-vanishing flux, the type I tachyons received a discrete shift in the mass, whereas for the type II tachyons the mass shift was continuous in the three-form flux. The question now arises what kind of instabilities these new tachyons indicate?

H-flux background

Let us discuss the H -flux background first. As mentioned in section 3.1, a flat background with a constant H -flux satisfies the 26-dimensional string equations of motion only up to linear order in H . Thus, we expect that fluctuations around this background will detect higher order corrections in H and will therefore develop tachyonic modes, i.e. relevant operators from the two-dimensional perspective, that will induce a renormalization group flow of the theory towards a truly conformal fixed point. One can immediately propose two candidate conformally invariant theories to which the unstable theory might flow:

- The theory should certainly be able to flow back to the flat background with vanishing H -flux.
- It is well known that for $H \neq 0$ there exists another conformally invariant theory, which is the $\widehat{su}(2)_k$ WZW model corresponding to a constant H -flux through a three-sphere S^3 .

It is now tempting to speculate that the two kinds of tachyons mentioned above correspond to these two conformal field theories. In this case, the type I tachyon should be identified with the $\widehat{su}(2)_k$ model and the type II tachyon with the trivial background.

R-flux background

Given the discussion for the H -flux background, we may ask about the analogous structure for the tachyons in the non-geometric R -flux background. The type II tachyon is again expected to lead to a flow towards the trivial theory with vanishing R -flux. However, the type I tachyon is expected to induce a flow towards a theory which is isomorphic to the $\widehat{su}(2)_k$ CFT. In view of the current algebra (3.17), a natural candidate may be provided by an *asymmetric* $\widehat{su}(2)_k$ model. The only difference to the usual symmetric $\widehat{su}(2)_k$ WZW model is a flip in the relative sign of the chiral and anti-chiral symmetry algebras. More concretely, with the plus sign standing for the symmetric $\widehat{su}(2)_k^{++}$ model and the minus sign denoting the asymmetric $\widehat{su}(2)_k^{+-}$ model, the chiral and anti-chiral

Kac-Moody algebras read

$$\begin{aligned} [j_m^a, j_n^b] &= +i f_c^{ab} j_{m+n}^c + k m \delta^{ab} \delta_{m+n} , \\ [\bar{j}_m^a, \bar{j}_n^b] &= \pm i f_c^{ab} \bar{j}_{m+n}^c + k m \delta^{ab} \delta_{m+n} . \end{aligned} \quad (5.1)$$

When constructing highest weights representations for these two theories, the only difference is that the anti-holomorphic raising and lowering operators are defined differently. In particular, we find

$$\begin{aligned} \widehat{su}(2)_k^{++} : \{ J_m^3, J_m^\pm = J_m^1 \pm i J_m^2 \} \times \{ \bar{J}_m^3, \bar{J}_m^\pm = \bar{J}_m^1 \pm i \bar{J}_m^2 \} , \\ \widehat{su}(2)_k^{+-} : \{ J_m^3, J_m^\pm = J_m^1 \pm i J_m^2 \} \times \{ \bar{J}_m^3, \bar{J}_m^\pm = \bar{J}_m^2 \pm i \bar{J}_m^1 \} . \end{aligned} \quad (5.2)$$

One therefore obtains the same representations, characters and modular invariant partition functions for both theories, so that on the level of the CFT the $\widehat{su}(2)_k^{++}$ and $\widehat{su}(2)_k^{+-}$ model are indistinguishable. However, even though at the level of the conformal field theory there is no difference between these two models, the target-space interpretation is different, namely one is geometric and the other is non-geometric. Our conjecture is that the non-trivial conformal fixed point of the model with R -flux is the asymmetric $\widehat{su}(2)_k^{+-}$ model. This would imply an intricate relationship between *R-flux* and *asymmetric string vacua*. We have to admit that in our situation the asymmetry is barely visible, but in more general cases it should be more apparent.

5.2 A tri-product

In the previous sections, we have computed scattering amplitudes of N tachyons and in the case of constant R -flux, have detected relative phase factors between different orders of insertion of the vertex operators. As required by conformal symmetry or crossing symmetry, respectively, (at linear order in θ) these phases became trivial after invoking momentum conservation. In this section, we show that these relative phase factors can be rephrased in terms of a generalization of the Moyal-Weyl star-product, which we call a tri-product.

In particular, the phase appearing in the three-point correlator (4.6) indicates that we can define a three-product of functions $f(x)$ in the following way⁹

$$f_1(x) \triangle f_2(x) \triangle f_3(x) \stackrel{\text{def}}{=} \exp\left(\frac{\pi^2}{2} \theta^{abc} \partial_a^{x_1} \partial_b^{x_2} \partial_c^{x_3}\right) f_1(x_1) f_2(x_2) f_3(x_3) \Big|_x , \quad (5.3)$$

where we used the notation $(\)|_x = (\)|_{x_1=x_2=x_3=x}$. Choosing $f_n(x) = \exp(ip_n \cdot x)$ we obtain formula (4.7), which after integration over x gives

$$\begin{aligned} \int d^3x f_1(x) \triangle f_2(x) \triangle f_3(x) &= \exp\left(-i \frac{\pi^2}{2} \theta^{abc} p_{1,a} p_{2,b} p_{3,c}\right) \delta(p_1 + p_2 + p_3) \\ &= \int d^3x f_1(x) f_2(x) f_3(x) . \end{aligned} \quad (5.4)$$

⁹As repeatedly emphasized, our methods are only reliable up to linear order in the flux parameter θ .

Note that (5.3) is precisely the three-product anticipated in [11]. Indeed, the three-bracket for the coordinates x^a can then be re-derived as the completely antisymmetrized sum of three-products

$$[x^a, x^b, x^c] = \sum_{\sigma \in P^3} \text{sign}(\sigma) x^{\sigma(a)} \Delta x^{\sigma(b)} \Delta x^{\sigma(c)} = 3\pi^2 \theta^{abc}, \quad (5.5)$$

where P^3 denotes the permutation group of three elements. In [11] this three-bracket was defined as the Jacobi-identity of the coordinates, which can only be non-zero if the space is noncommutative and nonassociative.

Next we consider the N -tachyon amplitude and the phase appearing in equation (4.13). This motivates us to define the N -product

$$f_1(x) \Delta_N f_2(x) \Delta_N \dots \Delta_N f_N(x) \stackrel{\text{def}}{=} \exp \left[\frac{\pi^2}{2} \theta^{abc} \sum_{1 \leq i < j < k \leq N} \partial_a^{x_i} \partial_b^{x_j} \partial_c^{x_k} \right] f_1(x_1) f_2(x_2) \dots f_N(x_N) \Big|_x, \quad (5.6)$$

which is the closed string generalization of the open string noncommutative product (2.5). This completely defines the new tri-product, which satisfies the relation

$$f_1 \Delta_N f_2 \Delta_N \dots \Delta_N f_{N-1} \Delta_N 1 = f_1 \Delta_{N-1} \dots \Delta_{N-1} f_{N-1}. \quad (5.7)$$

Specializing this expression to $N = 3$ gives

$$f_1 \Delta_2 f_2 = f_1 \Delta_3 f_2 \Delta_3 1 = f_1 \cdot f_2, \quad (5.8)$$

which just means that the tri-product of two functions is the usual commutative point-wise product. However, there are two main differences compared to the open string case.

- For the open string the star N -product was related to successive application of the usual Moyal-Weyl bi-product. This simplifying behavior is not true for the tri-product, i.e. the N -products Δ_N *cannot* be related to successive applications of the three-product $\Delta = \Delta_3$. For instance, for $N = 5$ we find

$$f_1 \Delta_5 f_2 \Delta_5 f_3 \Delta_5 f_4 \Delta_5 f_5 \neq [f_1 \Delta f_2 \Delta f_3] \Delta f_4 \Delta f_5. \quad (5.9)$$

- In contrast to the open string case, the effect of the tri-product in integrals vanishes, i.e.

$$\int d^n x f_1(x) \Delta_N f_2(x) \Delta_N \dots \Delta_N f_N(x) = \int d^n x f_1(x) f_2(x) \dots f_N(x). \quad (5.10)$$

In other words, the difference between the tri-product and the ordinary product is a total derivative.

The first item means that one does not only have to specify a deformed product of three functions (with the rest following), but has to specify a definition for a deformed product of any number of functions. We suspect that this is a very general behavior of such NCA geometries. The second item means that closed strings can consistently be defined on such nonassociative backgrounds, since in string scattering amplitudes its effect vanishes, i.e. *closed on-shell strings are blind against this deformation*.

Let us finish our discussion by mentioning that the mathematical analysis of such nonassociative spaces is beyond the scope of this paper.¹⁰ We also would like to refer the reader to figure 2 where the two proposals made in this section for T-duality and tachyon condensation of our initially constant H -flux on flat space configuration are illustrated.

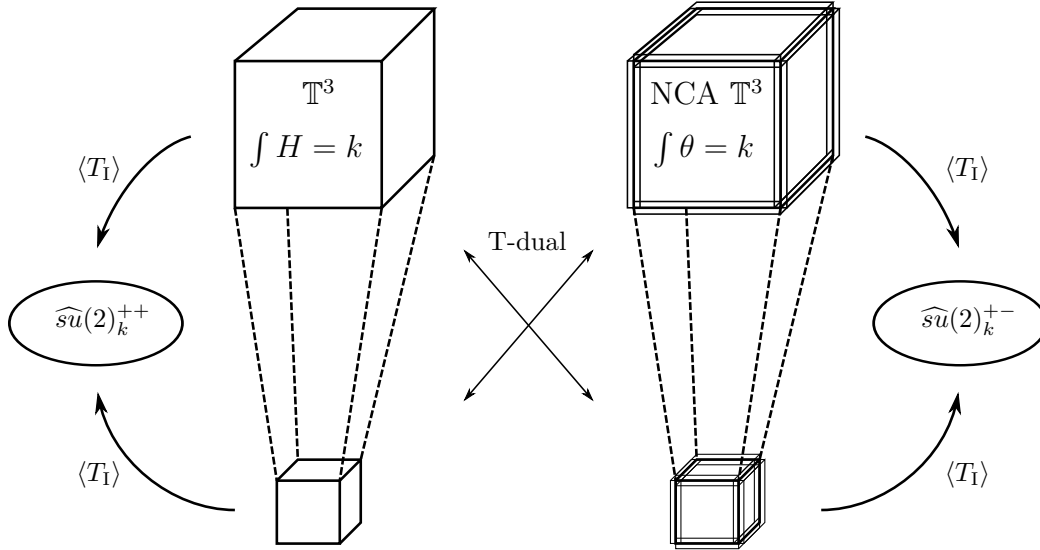


Figure 2: T-duality relation between a \mathbb{T}^3 -compactification with constant flux $\int_{T^3} H = k$ and a nonassociative \mathbb{T}^3 with constant R -flux $\int_{T^3} \theta = k$. Shown are also the proposed results of type I tachyon condensation, which do only depend on the quantized flux and not the initial size of the \mathbb{T}^3 .

6 Conclusions

In this paper, we have studied the structure of closed strings moving in three-form flux backgrounds. Our starting point was a flat space-time with constant H -flux, which via T-dualities is related to geometric and non-geometric flux backgrounds.

¹⁰It would be interesting to know whether this tri-product is compatible with the rather abstract notion of nonassociativity in [6].

In particular, the H - and R -flux configurations were investigated up to linear order in the flux in which they are expected to satisfy the string equations of motion.

Since our objection was to see whether a new nonassociative product for the R -flux background can appear, we followed an approach analogously to the constant B -form background in the open string case. We computed the three-point function of three coordinates \mathcal{X}^a via conformal perturbation theory and found that it can be expressed in terms of the Rogers dilogarithm. This result is consistent with what was derived as a limiting case of the $SU(2)$ WZW-model in [11]. Using this three-point function, we explicitly determined scattering amplitudes which revealed a much more intricate structure than in the open string case. The reason is that for the open string one can work with a free conformal field theory, whereas for the closed string the theory is interacting and only trustable (i.e. conformally invariant) up to linear order in the fluxes. We explicitly derived a number of OPEs and correlation functions of this CFT_H .

Furthermore, in the case of R -flux we found relative phases factors in the N -tachyon amplitudes upon permutation of two operators, which vanish after applying momentum conservation. We encoded the appearing phases via a new nonassociative tri-product, which generalizes the Moyal-Weyl product to closed strings with R -flux and supports our proposal that nonassociative spaces are relevant for these backgrounds. This analysis showed that on-shell such a NCA deformation of the target-space is compatible with the structure of two-dimensional conformal field theory. However, off-shell the nonassociative structure should become much more visible.

Moreover, we derived a conformally invariant and crossing symmetric four-tachyon amplitude and studied its pole structure. We observed the appearance of two new types of tachyons, for which we presented an interpretation in terms of the apparent instabilities of the system. In the case of R -flux we conjectured that the initial model flows towards an asymmetric version of the WZW model. Thus, non-geometric R -flux could be related to left-right asymmetric string backgrounds, which from the target space perspective seem to involve nonassociative geometries. We believe that this points towards a coherent picture for the appearance of non-standard geometric structures in string theory. Also, recall from [31] that for the open string a non-trivial two-form flux can be generated on a D-brane via an asymmetric rotation. As just discussed, asymmetric solutions (like in the case of R -flux) are similarly related to nonassociative bulk spaces probed by closed strings. From this perspective, commuting and associative geometries are rather the exception; generic solutions of string theory should involve NC-brane and NCA-bulk geometries.

The analysis of the CFT_H theory at linear order in flux presented in this paper clearly leads to a number of interesting issues worth to be studied. These include

- A generalization from the bosonic string investigated in this paper to the superstring, that is the construction of a SCFT_H .
- The OPEs for the graviton vertex operators show features reminiscent of logarithmic CFTs. It would be interesting to further analyze this structure.
- Furthermore, the construction of the boundary CFT_H is worth pursuing. In this case, the Freed-Witten anomaly [32] should be directly visible in the boundary states since this anomaly is an effect linear in the flux H .

We close with a puzzling question. In our approach we treated the non-geometric R -flux as a constant background. As we have illustrated, we expect that some of the gravitons at zero order in the flux become massive after taking into account corrections at linear order. But, *to what kind of fluctuation do the antisymmetric polarizations correspond to?* One possibility is that these are modes of the B -field, though they could also correspond to R -flux fluctuations implying changes in the NCA parameter θ . In the latter case, one would promote the constant parameter θ^{abc} to a field $\Theta^{abc}(X)$ so that the nonassociative geometry is defined by a non-vanishing three-bracket of the form

$$[X^a, X^b, X^c] = \Theta^{abc}(X) . \quad (6.1)$$

Conformal symmetry should then lead to on-shell equations of motions for $\Theta^{abc}(X)$.

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A Appendix

A.1 T-dual flux backgrounds

In this appendix, we review some aspects of backgrounds with H -, geometric, non-geometric and R -flux.

H-flux background

Following [33, 1, 2], let us start with a flat, rectangular three-torus parametrized by coordinates x^1, x^2, x^3 with metric

$$ds^2 = R_1^2 (dx^1)^2 + R_2^2 (dx^2)^2 + R_3^2 (dx^3)^2 . \quad (\text{A.1})$$

The radii of \mathbb{T}^3 are denoted by R_1, R_2, R_3 , and we allow for a constant H -flux such that

$$\int_{T^3} H = N \in \mathbb{Z} . \quad (\text{A.2})$$

This implies that the sigma-model equations of motion are satisfied only at linear order in H . We can therefore consider this configuration as a consistent string background only up to this order. Furthermore, we are free to choose a gauge in which the B -field reads

$$B_{x^2 x^3} = N x^1 . \quad (\text{A.3})$$

As it turns out, it is useful to consider the above three-torus as a \mathbb{T}^2 in the (x^2, x^3) direction fibered over an S^1 in the x^1 direction. This allows us to define a complex structure modulus τ and a Kähler modulus ρ for \mathbb{T}^2 as

$$\tau = i R_2 / R_3 , \quad \rho(x^1) = N x^1 + i R_2 R_3 , \quad (\text{A.4})$$

which encodes (A.2) as a parabolic monodromy

$$\rho \rightarrow \rho + 2\pi R_1 N \quad \text{when} \quad x^1 \rightarrow x^1 + 2\pi R_1 . \quad (\text{A.5})$$

The monodromy can be realized as a $SL(2, \mathbb{Z})$ transformation¹¹ on ρ , i.e. the moduli are preserved up to a $SL(2, \mathbb{Z})$ transformations when going once around the base circle.

¹¹To be more precise, we have to divide by an identification $2\pi R_1 \sim 1$. It is useful to think of each modulus as being an element of \mathbb{CP}^1 with $SL(2, \mathbb{Z})/\mathbb{Z}$ acting merely as matrix multiplication on the homogeneous coordinates.

Geometric flux background

Let us now perform a T-duality transformation along one of the three isometric directions.¹² Utilizing the Buscher rules, we can perform a first T-duality in, say, the x^3 direction to obtain

$$ds^2 = R_1^2 (dx^1)^2 + R_2^2 (dx^2)^2 + \frac{1}{R_3^2} (dx^3 + Nx^1 dx^2)^2, \quad B = 0. \quad (\text{A.6})$$

The complex structure and Kähler modulus are exchanged, that is $\tau' = \rho$ and $\rho' = \tau$, and we observe that the new metric receives an explicit dependence on the base-space coordinate x^1 . In order for this metric to be globally well-defined we have to restore the periodicity along the base direction by identifying

$$(x^1, x^2, x^3) \sim (x^1 + 2\pi R_1, x^2, x^3 - 2\pi R_1 N x^2), \quad (\text{A.7})$$

which defines a so-called twisted torus.

In order to interpret the role of N in this geometry, we introduce a dual basis of globally defined one-forms for $T(\mathbb{T}^3)$ as

$$\eta^1 = dx^1, \quad \eta^2 = dx^2, \quad \eta^3 = dx^3 + Nx^1 dx^2. \quad (\text{A.8})$$

Employing Cartan's structure equation for a torsion-free connection $\omega^a{}_b$

$$d\eta^a = \eta^b \wedge \omega^a{}_b, \quad (\text{A.9})$$

for the above basis the only non-vanishing component of $\omega^a{}_b$ turns out to be

$$\omega^{x^3}{}_{x^1 x^2} = -N, \quad (\text{A.10})$$

which is usually referred to as geometric flux. However, in general Cartan's structure equation (A.9) yields another constraint on $\omega^a{}_{bc}$ upon demanding $d^2\eta^a = 0$. The resulting Jacobi identity $\omega^a{}_{b[c}\omega^b{}_{de]} = 0$ implies that we can consider $\omega^a{}_{bc}$ to be structure constants of the Lie algebra. Furthermore, for compact spaces one has to require in addition that $\omega^a{}_{ab} = 0$ (no sum) which is satisfied by nilpotent algebras, and the resulting manifolds are called nilmanifolds.

Non-geometric flux background

Since the metric (A.6) does not depend on x^2 explicitly, we are allowed to perform a second T-duality in the remaining direction of \mathbb{T}^2 . Applying the Buscher rules once more yields

$$ds^2 = R_1^2 (dx^1)^2 + \frac{1}{R_2^2 R_3^2 + N^2 (x^1)^2} (R_3^2 (dx^2)^2 + R_2^2 (dx^3)^2), \quad (\text{A.11})$$

$$B_{x^2 x^3} = -\frac{Nx^1}{R_2^2 R_3^2 + N^2 (x^1)^2}.$$

¹²We should mention that [34] describes a method for T-dualizing without referring to any Killing symmetries.

The complex structure and Kähler modulus can be computed to be

$$\tau'' = i R_3/R_2 , \quad \rho'' = -1/(N x^1 + i R_2 R_3) , \quad (\text{A.12})$$

leading to the monodromy

$$1/\rho'' \rightarrow 1/\rho'' + 2\pi R_1 N \quad \text{when} \quad x^1 \rightarrow x^1 + 2\pi R_1 . \quad (\text{A.13})$$

This can still be realized as a parabolic $SL(2, \mathbb{Z})_\rho$ transformation. However, although the metric and the B -field are well-defined locally, it is not possible to describe them globally. The transition functions between local trivializations mix the B -field with the metric on the total space, which is known as a T -fold [5]. Note that in the present case, the parameter N is related to the so-called non-geometric flux $Q_{x^1 x^2 x^3} = N$.

R-flux background

Finally, we can consider a T-duality along the base direction x^1 which is however not captured by the Buscher rules as (A.11) does not admit an isometry in this direction. But, in [1, 2] it is argued that although the background obtained by performing another T-duality seems to elude a geometric description even locally, it has to be included in a background independent formulation of string theory. We characterize this background by a new type of flux, denoted by $R^{x^1 x^2 x^3} = N$. It is obtained by formally applying the following chain of three T-duality transformations

$$H_{x^1 x^2 x^3} \xleftrightarrow{T_{x^3}} \omega_{x^1 x^2 x^3} \xleftrightarrow{T_{x^2}} Q_{x^1 x^2 x^3} \xleftrightarrow{T_{x^1}} R^{x^1 x^2 x^3} . \quad (\text{A.14})$$

A.2 The Rogers dilogarithm

In this appendix, we summarize some properties of the complex Rogers dilogarithm and recall one of its generalizations defined in [35]. The latter will be used in the main text to rewrite the four-tachyon amplitude (4.14) in a form which only depends on the cross-ratio

$$z = \frac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_3)(z_2 - z_4)} , \quad (\text{A.15})$$

implying that this amplitude is manifestly invariant under $SL(2, \mathbb{C})$ transformations. For a more detailed analysis of the mathematical aspects of the Rogers dilogarithm function we would like to refer the reader to [36, 37], whereas its generalization is described in detail in [35, 38].

Definition and fundamental properties

The Rogers dilogarithm function $L(x)$ for real arguments x is defined in the following way

$$L(x) := \text{Li}_2(x) + \frac{1}{2} \log(x) \log(1-x) , \quad 0 < x < 1 , \quad (\text{A.16})$$

where $\text{Li}_2(x)$ denotes the Euler dilogarithm function given by

$$\text{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} = - \int_0^x \frac{\log(1-y)}{y} , \quad 0 \leq x \leq 1 . \quad (\text{A.17})$$

With the help of (A.17), the integral representation of the Rogers dilogarithm can be deduced as

$$L(x) = -\frac{1}{2} \int_0^x \left(\frac{\log(1-y)}{y} + \frac{\log(y)}{1-y} \right) dy . \quad (\text{A.18})$$

Furthermore, from these definitions one can derive two functional relations, which in turn uniquely characterize the Rogers dilogarithm function

$$\begin{aligned} L(x) + L(1-x) &= L(1) , \\ L(x) - L(y) + L\left(\frac{y}{x}\right) - L\left(\frac{1-x^{-1}}{1-y^{-1}}\right) + L\left(\frac{1-x}{1-y}\right) &= 0 . \end{aligned} \quad (\text{A.19})$$

Employing the integral representation (A.18), one can analytically continue $L(x)$ to the domain $\mathbb{C} \setminus \{0, 1\}$. However, the resulting function $L(z)$ is not single valued any more and one should use the universal cover of $\mathbb{C} \setminus \{0, 1\}$ as the domain of definition. For the complex Rogers dilogarithm the relation

$$L(z) + L(1-z) = L(1) \quad (\text{A.20})$$

still holds, but the five-term relation in (A.19) receives logarithmic corrections. To describe the systematics of those corrections, let us introduce the following generalization which is due to Neumann [35]

$$R(z; p, q) := L(z) - \frac{\pi^2}{6} + \frac{\pi i}{2} \left(p \log(z-1) + q \log z \right) . \quad (\text{A.21})$$

Here, p, q are integer numbers and the constant is just a convenient normalization. For (A.21) it is then possible to establish again a five-term relation

$$\begin{aligned} R(x; p_0, q_0) - R(y; p_1, q_1) + R\left(\frac{y}{x}; p_2, q_2\right) \\ - R\left(\frac{1-x^{-1}}{1-y^{-1}}; p_3, q_3\right) + R\left(\frac{1-x}{1-y}; p_4, q_4\right) = 0 , \end{aligned} \quad (\text{A.22})$$

where the integers p_i, q_i have to obey some restrictions called *flattening conditions*.

Five-term relation

In the following, we sketch a geometric viewpoint on the arguments of the five-term relation (A.22), but refer the reader to the original articles [35, 38] for a more comprehensive treatment.

Let $z_1 \dots z_5$ be five distinct points in $\mathbb{C} \cup \{\infty\}$. If we choose four of them, we can interpret them as endpoints of a tetrahedron in hyperbolic three-space \mathbb{H}^3 located at $\partial\mathbb{H}^3 = \mathbb{CP}^1$, which is also known as an *ideal* tetrahedron. Up to congruence, such an object is characterized by the cross-ratio of its four endpoints. For instance, omitting z_5 we have

$$[z_1 : z_2 : z_3 : z_4] := \frac{z_{32} z_{41}}{z_{31} z_{42}} . \quad (\text{A.23})$$

Now we observe that, if we omit from $z_1 \dots z_5$ one point we get cross-ratio parameters corresponding to the arguments appearing in the five term relation (A.22)

$$\begin{aligned} [z_2 : z_3 : z_4 : z_5] &=: x , & [z_1 : z_3 : z_4 : z_5] &=: y , \\ [z_1 : z_2 : z_4 : z_5] &= \frac{y}{x} , & [z_1 : z_2 : z_3 : z_5] &= \frac{1 - x^{-1}}{1 - y^{-1}} , \\ [z_1 : z_2 : z_3 : z_4] &= \frac{1 - x}{1 - y} . \end{aligned} \quad (\text{A.24})$$

This suggests a geometric reason for the five-term relation. Indeed, if we consider the so called scissors congruence group $\mathcal{P}(\mathbb{H}^3)$, which is given by the free \mathbb{Z} -module generated by three-dimensional polytopes in \mathbb{H}^3 modulo congruence relations (i.e. if we denote by $[P]$ the class of a polytope P , then we have $[P] = [P_1] + \dots + [P_n]$, if we get P by gluing P_1, \dots, P_n along common faces), it turns out that in $\mathcal{P}(\mathbb{H}^3)$ we obtain the following relation

$$[x] + \left[\frac{y}{x}\right] + \left[\frac{1-x}{1-y}\right] = [y] + \left[\frac{1-x^{-1}}{1-y^{-1}}\right] , \quad (\text{A.25})$$

where $[z]$ denotes the class of the tetrahedron with cross-ratio parameter z . Now, as stated in proposition 4.5 of [35], it turns out that a similar relation holds also on the domain of definition of the extended version (A.21) of the Rogers dilogarithm, if the additional parameters obey the so-called flattening condition. For instance, if all the arguments are in the upper complex half-plane this condition is given by the following set of linear equations

$$\begin{aligned} p_2 &= p_1 - p_0 , & p_3 &= p_1 - p_0 + q_1 - q_0 , \\ p_4 &= q_1 - q_0 , & q_3 &= q_2 - q_1 , \\ q_4 &= q_2 - q_1 - p_0 . \end{aligned} \quad (\text{A.26})$$

For general positions of the arguments, we refer the reader to [38].

Details on the derivation of the four-tachyon correlator

We now want to apply this formalism to the four-tachyon correlator shown in equation (4.14), where in the holomorphic sector the combination $\Xi(z_i) = \mathcal{L}(\frac{z_{12}}{z_{13}}) - \mathcal{L}(\frac{z_{12}}{z_{14}}) + \mathcal{L}(\frac{z_{13}}{z_{14}}) - \mathcal{L}(\frac{z_{23}}{z_{24}})$ has appeared. Writing out the \mathcal{L} function in terms of the Rogers dilogarithm L , we find

$$\begin{aligned}\Xi(z_i) = & L\left(\frac{z_{12}}{z_{13}}\right) - L\left(\frac{z_{12}}{z_{14}}\right) + L\left(\frac{z_{13}}{z_{14}}\right) - L\left(\frac{z_{23}}{z_{24}}\right) \\ & + L\left(\frac{z_{13}}{z_{23}}\right) - L\left(\frac{z_{14}}{z_{24}}\right) + L\left(\frac{z_{14}}{z_{34}}\right) - L\left(\frac{z_{24}}{z_{34}}\right) \\ & + L\left(\frac{z_{32}}{z_{12}}\right) - L\left(\frac{z_{42}}{z_{12}}\right) + L\left(\frac{z_{43}}{z_{13}}\right) - L\left(\frac{z_{43}}{z_{23}}\right).\end{aligned}\tag{A.27}$$

Let us then take as above five points in $\mathbb{C} \cup \infty$, which are chosen as $z_1 \dots z_4$ and the point ∞ . Recall that our eventual goal is to use the five-term relation in order to express the correlator as

$$\Xi(z_i) \simeq L(z) + L\left(\frac{1}{1-z}\right) + L\left(1 - \frac{1}{z}\right) + C, \tag{A.28}$$

where z is the $SL(2, \mathbb{C})$ invariant cross-ratio (A.15) and C is a constant to be determined later.

We proceed by working backwards and writing down the cross-ratios (by omitting every vertex once) corresponding to different permutations of the vertices $z_1, z_2, z_3, \infty, z_4$, which gives the desired arguments in (A.28). According to the above sketched formalism, the other cross-ratios will then have the characteristic form needed for the application of the five-term relation. More concretely, we start with the vertices $z_1, z_2, z_3, \infty, z_4$ leading to

$$\begin{aligned}[z_1 : z_2 : z_3 : \infty] &= \frac{z_{32}}{z_{31}} =: x_0^A, & [z_2 : z_3 : \infty : z_4] &= \frac{z_{42}}{z_{43}} =: x_1^A, \\ [z_1 : z_3 : \infty : z_4] &= \frac{z_{41}}{z_{43}} =: x_2^A, & [z_1 : z_2 : \infty : z_4] &= \frac{z_{41}}{z_{42}} =: x_3^A, \\ [z_1 : z_2 : z_3 : z_4] &= \frac{z_{32} z_{41}}{z_{31} z_{42}} =: x_4^A.\end{aligned}\tag{A.29}$$

Note that the cross ratio x_4^A is precisely the cross ratio in (A.15). Moreover, one finds $\frac{x_4^A}{x_0^A} = x_3^A$ and the remaining two characteristic ratios appearing in the five-term relation directly yield x_1^A and x_2^A . However, only four out of the twelve terms in (A.27) have been taken care off so far. For the other eight, let us first permute the vertices to $z_1, z_4, z_2, \infty, z_3$ and follow the same procedure as before.

We obtain the following ratios

$$\begin{aligned}
[z_1 : z_4 : z_2 : \infty] &= \frac{z_{24}}{z_{21}} =: x_0^B, & [z_4 : z_2 : \infty : z_3] &= \frac{z_{34}}{z_{32}} =: x_1^B, \\
[z_1 : z_2 : \infty : z_3] &= \frac{z_{31}}{z_{32}} =: x_2^B, & [z_1 : z_4 : \infty : z_3] &= \frac{z_{31}}{z_{34}} =: x_3^B, \\
[z_1 : z_4 : z_2 : z_3] &= \frac{z_{24} z_{31}}{z_{21} z_{34}} =: x_4^B.
\end{aligned} \tag{A.30}$$

Note again, the ratio omitting ∞ gives us the argument $\frac{1}{1-z}$. In the final case we permute the vertices to $z_1, z_3, z_4, \infty, z_2$ to find

$$\begin{aligned}
[z_1 : z_3 : z_4 : \infty] &= \frac{z_{43}}{z_{41}} =: x_0^C, & [z_3 : z_4 : \infty : z_2] &= \frac{z_{23}}{z_{24}} =: x_1^C, \\
[z_1 : z_4 : \infty : z_2] &= \frac{z_{21}}{z_{24}} =: x_2^C, & [z_1 : z_3 : \infty : z_2] &= \frac{z_{21}}{z_{23}} =: x_3^C, \\
[z_1 : z_3 : z_4 : z_2] &= \frac{z_{43} z_{21}}{z_{41} z_{23}} =: x_4^C,
\end{aligned} \tag{A.31}$$

and the ratio omitting ∞ is the desired argument $1 - \frac{1}{z}$. Comparing the above cross-ratios with the arguments in the four-point correlator (A.27), we observe that we can rewrite it in terms of the variables x_k^L defined above

$$\begin{aligned}
\Xi(z_i) &= L(1 - x_0^A) - L\left(-\frac{x_2^C}{1-x_2^C}\right) + L(1 - x_0^C) - L(x_1^C) \\
&\quad + L(x_2^B) - L(x_3^A) + L(x_2^A) - L(x_1^A) \\
&\quad + L\left(\frac{1}{x_3^C}\right) - L(x_0^B) + L\left(\frac{1}{x_3^B}\right) - L(x_1^B).
\end{aligned} \tag{A.32}$$

To make contact with the extended Rogers dilogarithm and its five-term relation, we recall the result of the basic three-point function

$$\langle \mathcal{X}^a(z_1, \bar{z}_1) \mathcal{X}^b(z_2, \bar{z}_2) \mathcal{X}^c(z_3, \bar{z}_3) \rangle = \theta^{abc} \left[L\left(\frac{z_{12}}{z_{13}}\right) + L\left(\frac{z_{23}}{z_{21}}\right) + L\left(\frac{z_{13}}{z_{23}}\right) \right] + F(z_1, z_2, z_3) - \text{c.c.}, \tag{A.33}$$

where $F(z_1, z_2, z_3)$ denotes integration “constants,” whose third mixed derivative vanishes. We then observe that the extended Rogers dilogarithm

$$R\left(\frac{z_{ij}}{z_{ik}}\right) = L\left(\frac{z_{ij}}{z_{ik}}\right) + \frac{i\pi}{2} \left[p \log\left(\frac{z_{ij}}{z_{ik}}\right) + q \log\left(\frac{z_{jk}}{z_{ik}}\right) \right] \tag{A.34}$$

precisely provides only correction terms which can be interpreted as such integration constants $F(z_1, z_2, z_3)$. Thus, taking this choice into account, we can

introduce the extended Rogers dilogarithm in (A.32). After some reordering we obtain

$$\begin{aligned} \Xi(z_i) = & R(1 - x_0^A; p_0^A, q_0^A) - R(x_1^A; p_1^A, q_1^A) + R(x_2^A; p_2^A, q_2^A) - R(x_3^A; p_3^A, q_3^A) \\ & - R(x_0^B; p_0^B, q_0^B) - R(x_1^B; p_1^B, q_1^B) + R(x_2^B; p_2^B, q_2^B) + R\left(\frac{1}{x_3^B}; p_3^B, q_3^B\right) \\ & + R(1 - x_0^C; p_0^C, q_0^C) - R(x_1^C; p_1^C, q_1^C) - R\left(-\frac{x_2^C}{1-x_2^C}; p_2^C, q_2^C\right) + R\left(\frac{1}{x_3^C}; p_3^C, q_3^C\right). \end{aligned} \quad (\text{A.35})$$

Note that for the extended Rogers dilogarithm the following transformation formulas can be applied [39]

$$\begin{aligned} R(1 - x; p, q) &= -R(x; -p, p + q - \epsilon) - \frac{\pi^2}{6}, \\ R\left(\frac{1}{x}; p, q\right) &= -R(x; -p, p + q - \epsilon) - p\frac{\pi^2}{2}, \\ R\left(-\frac{x}{1-x}; p, q\right) &= -R(x; p + q - \epsilon, -q) - \frac{\pi^2}{3} + q\frac{\pi^2}{2}, \end{aligned} \quad (\text{A.36})$$

where $\epsilon = \pm 1$ for x in the upper or lower complex half plane, respectively. It is therefore possible to rewrite the dilogarithm terms purely in terms of the cross ratio parameters determined in (A.29)-(A.31). If we also take care of the relations between those cross-ratios, up to a constant term we find

$$\begin{aligned} \Xi(z_i) = & -R(x_0^A; -p_0^A, p_0^A + q_0^A - \epsilon) - R\left(\frac{1-x_0^A}{1-x_4^A}; p_1^A, q_1^A\right) \\ & + R\left(\frac{1-(x_0^A)^{-1}}{1-(x_4^A)^{-1}}; p_2^A, q_2^A\right) - R\left(\frac{x_4^A}{x_0^A}; p_3^A, q_3^A\right) \\ & - R(x_0^B; p_0^B, q_0^B) - R\left(\frac{1-x_0^B}{1-x_4^B}; p_1^B, q_1^B\right) \\ & + R\left(\frac{1-(x_0^B)^{-1}}{1-(x_4^B)^{-1}}; p_2^B, q_2^B\right) - R\left(\frac{x_4^B}{x_0^B}; -p_3^B, p_3^B + q_3^B - \epsilon\right) \\ & - R(x_0^C; -p_0^C, p_0^C + q_0^C - \epsilon) - R\left(\frac{1-x_0^C}{1-x_4^C}; p_1^C, q_1^C\right) \\ & + R\left(\frac{1-(x_0^C)^{-1}}{1-(x_4^C)^{-1}}; -p_2^C, p_2^C + q_2^C - \epsilon\right) - R\left(\frac{x_4^C}{x_0^C}; p_3^C + q_3^C - \epsilon, -q_3^C\right). \end{aligned} \quad (\text{A.37})$$

Imposing now the flattening conditions separately on $p_i^{A,B,C}, q_j^{A,B,C}$ in such a way that

$$p_4^{A,B,C} = q_4^{A,B,C} = 0, \quad (\text{A.38})$$

(this is possible since in every region the flattening condition determines only five of the 10 parameters involved in the five-term relation in terms of the remaining

ones), we finally get

$$\Xi(z_i) = -R(x_4^A; 0, 0) - R(x_4^B; 0, 0) - R(x_4^C; 0, 0) + C' = -\mathcal{L}(z) + C' , \quad (\text{A.39})$$

which is equal to (A.28) up to a constant. We can now uniquely fix this constant by requiring crossing symmetry of the holomorphic part of the four-point function. Due to the fundamental relation (A.20), one obtains $C' = \frac{3}{2}L(1) = \frac{\pi^2}{4}$. To summarize, employing the five-term relation for the extended Rogers dilogarithm, it is possible to chose the integration constants in our basic correlator such that

$$\mathcal{L}\left(\frac{z_{12}}{z_{13}}\right) - \mathcal{L}\left(\frac{z_{12}}{z_{14}}\right) + \mathcal{L}\left(\frac{z_{13}}{z_{14}}\right) - \mathcal{L}\left(\frac{z_{23}}{z_{24}}\right) = -\frac{3}{2}L(1) + \mathcal{L}(X) , \quad (\text{A.40})$$

so that the four point tachyon correlator is both crossing symmetric and depends only on the cross ratio $X = 1 - z$ (and is therefore $SL(2, \mathbb{C})$ -invariant).

Relation between $\mathcal{L}(z)$ and $L(z)$

Let us finally elaborate on the relation between the \mathcal{L} function and the usual Rogers dilogarithm L . For the Taylor expansion of $\mathcal{L}(z)$ around $z = 0$ it is useful to first relate the sum in $\mathcal{L}(z)$ to $L(z)$. We will show that the following relation holds

$$L(z) + L\left(1 - \frac{1}{z}\right) + L\left(\frac{1}{1 - z}\right) = 3L(z) \pm \frac{i\pi}{2} \log(z(1 - z)) . \quad (\text{A.41})$$

Note that here $z \in \mathbb{C} \setminus \mathbb{R}$ and the plus sign holds for $\text{Im}(z) < 0$ while the minus sign holds for $\text{Im}(z) > 0$. For the proof of (A.41) we recall from equation (A.17) that the arguments of the Euler dilogarithm transform as

$$\begin{aligned} \text{Li}_2\left(\frac{z-1}{z}\right) &= \text{Li}_2(z) - \frac{\pi^2}{6} - \frac{1}{2} \log^2(z) + \log(1-z) \log(z) , \\ \text{Li}_2\left(\frac{1}{1-z}\right) &= \text{Li}_2(z) + \frac{\pi^2}{6} + \log(-z) \log(1-z) - \frac{1}{2} \log^2(1-z) . \end{aligned} \quad (\text{A.42})$$

We therefore find

$$\begin{aligned} L(z) + L\left(1 - \frac{1}{z}\right) + L\left(\frac{1}{1 - z}\right) \\ = 3L(z) + \frac{1}{2} \log(-z) \log(1-z) - \frac{1}{2} \log(z-1) \log(z) . \end{aligned} \quad (\text{A.43})$$

To rewrite the logarithmic terms with negative argument, one has to distinguish two cases depending on the sign of the imaginary part of z

$$\begin{aligned} \log(-z) &= \log(z) - i\pi & \text{for } \text{Im}(z) > 0 , \\ \log(-z) &= \log(z) + i\pi & \text{for } \text{Im}(z) < 0 . \end{aligned} \quad (\text{A.44})$$

Using this distinction in equation (A.43), one arrives at the desired result (A.41).

A.3 Details on correlation functions

In this appendix, we present some details on the computation of correlation functions which have appeared in the main text.

Correction to two-point function

We consider the correction to the two-point function of two fields $X^a(z, \bar{z})$ at second order in the H -flux, and for convenience we recall formula (3.28)

$$\begin{aligned} \delta_2 \langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) \rangle &= \frac{1}{2(6\pi\alpha')^2} H_{mno} H_{pqr} \int_{\Sigma} d^2 w_1 \int_{\Sigma} d^2 w_2 \\ &\quad \langle : X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) : : X^m(w_1, \bar{w}_1) \partial X^n(w_1) \bar{\partial} X^o(\bar{w}_1) : \\ &\quad : X^p(w_2, \bar{w}_2) \partial X^q(w_2) \bar{\partial} X^r(\bar{w}_2) : \rangle_0. \end{aligned} \quad (\text{A.45})$$

The correlator is evaluated using Wick contractions and can be simplified using partial integration. For the latter one should note that the integrals have to be regularized by cutting out of the integration region a small disc of radius $\epsilon \ll 1$ specified by $|\omega_1 - \omega_2| < \epsilon$. One is then left with computing

$$\begin{aligned} \delta_2 \langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) \rangle &= -\frac{\alpha'^2}{64\pi^2} H^a{}_{pq} H^{bpq} \int d^2 w_1 d^2 w_2 \frac{1}{z_1 - w_1} \frac{1}{\bar{z}_2 - \bar{w}_2} \frac{1}{|w_1 - w_2|^2}. \end{aligned} \quad (\text{A.46})$$

Let us now rewrite this expression in order to apply a variant of the inhomogeneous Cauchy formula. In particular, we have

$$\begin{aligned} \delta_2 \langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) \rangle &= -\frac{\alpha'^2}{64\pi^2} H^a{}_{pq} H^{bpq} \int d^2 w_2 \frac{1}{\bar{z}_2 - \bar{w}_2} \int d^2 w_1 \frac{1}{w_1 - w_2} \bar{\partial}_{\bar{w}_1} \left(\frac{\log |w_1 - w_2|^2}{z_1 - w_1} \right). \end{aligned} \quad (\text{A.47})$$

For the second integral we apply the regularization procedure mentioned above and remove the disc $D_{\epsilon}(w_2)$ of radius $\epsilon \ll 1$ from the w_1 -plane. More concretely, we compute

$$\begin{aligned} \int d^2 w_1 \frac{1}{w_1 - w_2} \bar{\partial}_{\bar{w}_1} \left(\frac{\log |w_1 - w_2|^2}{z_1 - w_1} \right) &= \oint_{\partial D_{\epsilon}(w_2)} dw_1 \frac{1}{w_1 - w_2} \frac{\log |w_1 - w_2|^2}{z_1 - w_1} \\ &= 4\pi i \frac{\log \epsilon}{z_1 - w_2}. \end{aligned} \quad (\text{A.48})$$

Using this result in (A.47), we obtain the expression

$$\delta_2 \langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) \rangle = -\frac{i \alpha'^2 \log \epsilon}{16\pi} H^a{}_{pq} H^{bpq} \int d^2 w_2 \frac{1}{z_1 - w_2} \frac{1}{\bar{z}_2 - \bar{w}_2}, \quad (\text{A.49})$$

for which we apply the above procedure once more. This leads to the final expression

$$\delta_2 \langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) \rangle = \frac{\alpha'^2}{8} H^a{}_{pq} H^{bpq} \log |z_1 - z_2|^2 \log \epsilon . \quad (\text{A.50})$$

N -tachyon correlator

We now turn to the evaluation of a correlator involving N tachyon vertex operators of the form

$$\mathcal{V}_i^- \equiv \mathcal{V}_{p_i}(z_i, \bar{z}_i) =: \exp(i p_i \cdot \mathcal{X}(z_i, \bar{z}_i)) : , \quad (\text{A.51})$$

where $i = 1, \dots, N$ labels the different operators and where we again employ the short-hand notation $p \cdot \mathcal{X} = p_a \mathcal{X}^a$. Using formula (3.11), up to first order in the flux the N -tachyon amplitude in the H -flux background can be expanded as

$$\langle \mathcal{V}_1 \dots \mathcal{V}_N \rangle^- = \langle \mathcal{V}_1 \dots \mathcal{V}_N \rangle_0^- - \langle \mathcal{V}_1 \dots \mathcal{V}_N \mathcal{S}_1 \rangle_0^- + \mathcal{O}(H^2) . \quad (\text{A.52})$$

Let us consider the second term involving \mathcal{S}_1 in more detail. Since the perturbation \mathcal{S}_1 is already linear in the flux H , we can replace the perturbed vertex operators \mathcal{V}_i^- by the ones of the free theory V_i^- , defined in terms of the free field $X^a(z, \bar{z})$. In this case, we can employ standard techniques to evaluate the correlator and find (at linear order in the flux)

$$\begin{aligned} \langle \mathcal{V}_1 \dots \mathcal{V}_N (-\mathcal{S}_1) \rangle_0^- &= \langle V_1 \dots V_N \rangle_0^- \times \\ &\sum_{1 \leq i < j < k \leq N} (-i) p_{i,a} p_{j,b} p_{k,c} \langle X^a(z_i, \bar{z}_i) X^b(z_j, \bar{z}_j) X^c(z_k, \bar{z}_k) \rangle^- . \end{aligned} \quad (\text{A.53})$$

Concerning the first term in (A.52), we recall the definition of the corrected field as $\mathcal{X}^a(z, \bar{z}) = X^a(z, \bar{z}) + \frac{1}{2} H^a{}_{bc} X_L^b(z) X_R^c(\bar{z})$, and expand the exponentials in the vertex operators up to linear order in H . Combining this contribution with (A.53) leads to the following result (up to linear order in the flux)

$$\begin{aligned} \langle \mathcal{V}_1 \dots \mathcal{V}_N \rangle^- &= \langle V_1 \dots V_N \rangle_0^- \left[1 + \sum_{1 \leq i < j < k \leq N} (-i) p_{i,a} p_{j,b} p_{k,c} \times \right. \\ &\quad \left. \langle \mathcal{X}^a(z_i, \bar{z}_i) \mathcal{X}^b(z_j, \bar{z}_j) \mathcal{X}^c(z_k, \bar{z}_k) \rangle^- \right] \\ &= \langle V_1 \dots V_N \rangle_0^- \left[1 - i \theta^{abc} \sum_{1 \leq i < j < k \leq N} p_{i,a} p_{j,b} p_{k,c} \left[\mathcal{L}\left(\frac{z_{ij}}{z_{ik}}\right) - \mathcal{L}\left(\frac{\bar{z}_{ij}}{\bar{z}_{ik}}\right) \right] \right] , \end{aligned} \quad (\text{A.54})$$

where we inserted the expression for the basic three-point function (3.26). The computation for vertex operators describing winding states \mathcal{V}_i^+ is analogous and

only leads to the usual sign flip between the holomorphic and anti-holomorphic contributions. With $[\dots]_\theta$ indicating that the exponential is to be expanded only up to linear order in θ , we then arrive at

$$\langle \mathcal{V}_1 \dots \mathcal{V}_N^\mp \rangle = \langle V_1 \dots V_N \rangle_0^\mp \exp \left[-i\theta^{abc} \sum_{1 \leq i < j < k \leq N} p_{i,a} p_{j,b} p_{k,c} \left[\mathcal{L}\left(\frac{z_{ij}}{z_{ik}}\right) \mp \mathcal{L}\left(\frac{\bar{z}_{ij}}{\bar{z}_{ik}}\right) \right] \right]_\theta . \quad (\text{A.55})$$

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